International Journal of Approximate Reasoning 124 (2020) 1–26

Contents lists available at ScienceDirect

International Journal of Approximate Reasoning

www.elsevier.com/locate/ijar

Double-quantitative variable consistency dominance-based rough set approach

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A R T I C L E I N F O

Article history:
Received 26 April 2019
Received in revised form 7 May 2020
Accepted 12 May 2020
Available online 1 June 2020

Keywords:
Dominance-based rough set
Double quantification
Ordered information system
Quantitative consistency
Three-way decision

A B S T R A C T

Rough set model with double quantification satisfies the requirement of quantitative information in practical applications, it has better fault tolerance than probabilistic rough set model considering only relative quantification and graded rough set model considering only absolute quantification. In this paper, two kinds of consistency levels are introduced from the perspective of double quantification in an ordered information system, namely relative quantitative consistency level and absolute quantitative consistency level. The single-quantitative variable consistency dominance-based rough set models based on these two kinds of quantitative consistency levels and their basic properties with the relevant three-way decision rules are discussed respectively in an ordered information system. Moreover, two kinds of double-quantitative variable consistency dominance-based rough set models and their basic properties with the relevant decision rules based on these two kinds of quantitative consistency levels are introduced. A consistency analysis of decision making in a practical case study is used to illustrate and interpret the double-quantitative variable consistency rough set models and the related decision rules in the ordered information system. The obvious shortcomings of dominance-based rough set approach (DRSA) without quantitative information are compared to explain the advantages of the quantitative variable consistency dominance-based rough sets with the two consistency levels in the practical case study.

1. Introduction

Rough set theory proposed by Pawlak [39], is an extension of the classical set theory and could be regarded as a mathematical and soft computing tool to handle imprecision, vagueness and uncertainty in data analysis. A key notion of rough set theory is the approximation of a basic set by a pair of definable sets called lower and upper approximations. It is characterized by a zero tolerance of errors, namely, an object in the lower approximation which certainly belongs to and an object in the complement of upper approximation which certainly does not belong to the set. Rough set models with quantitative information can be used to overcome certain limitation of Pawlak rough set and its generalized models [29,55]. The limitation indicates that Pawlak rough set and its relevant generalizations can not deal well with quantitative problems

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https://doi.org/10.1016/j.ijar.2020.05.002
0888-613X/

in real-life applications. The relationship between equivalence classes and the basic set is so strict that there are no fault tolerance mechanisms available, and the quantitative information about the degree of overlap of the equivalence classes and the basic set is not taken into consideration [54]. In fact, there are some degrees of inclusion relation between sets, and the extent of overlap of sets is important information to consider in applications.

Improving the Pawlak rough set model with quantitative information is a promising direction and expansions of the model that include such quantification are of particular relevance [69]. By introducing certain levels of errors, probabilistic rough set (PRS) [55,59] and graded rough set (GRS) [58] are basic quantitative generalizations of Pawlak rough set. PRS and GRS are two fundamental expansion models that achieve strong fault tolerance capabilities by utilizing quantitative descriptions [69]. At present, there are some related generalized forms of these two models being studied [22,23,34,36,37,46,56,63,70]. In order to have a better fault-tolerance mechanism, Greco et al. presented a parameterized rough set model by considering the relative and absolute rough memberships in the lower approximation and upper approximation [10], and this parameterized rough set model was recalled by Yao et al. in reference [57]. The main idea referred to literature [10] is to improve the fault-tolerance of rough approximations by considering two parameters in a model. However, in fact, it is not difficult to reveal that both relative rough membership and absolute rough membership considered in literature [10] belong to relative quantitative information essentially, and there is no absolute quantitative information involved. As we know, the relative and absolute measures reflect relative accuracy and absolute accuracy from two different quantitative viewpoints. Relative quantitative information and absolute quantitative information are two kinds of quantification methodologies encountered in certain applications [31,32,68]. Double quantification regarding their fusion has visible semantic background and feasibility. For this purpose, several works related to the double quantification have been explored [6,14,26,29,30,32,44,50,60,64–67].

As an important notion in rough set theory, the information system provides a convenient basis for the representation of objects in terms of their attributes. Due to the existence of complexity and uncertainty, several extensions of rough set model have been proposed for different requirements to deal with particular problems. The existing extended research on rough set models can be roughly summarized into two perspectives: (1) *Extending data type*. In Pawlak rough set model, the data type is classical, that is, each object in the information system has only one definite value with regard to each attribute. There are various generalized rough set models presented to tackle with the related data types in different information systems, which including set-valued data, interval-valued data, fuzzy data, intuitionistic fuzzy data, etc. And the corresponding rough set models are respectively called set-valued rough set [40], interval-valued rough set [15,17,45], fuzzy rough set (rough fuzzy set) [6,29,46], and intuitionistic fuzzy rough set [16,17]. There are also incomplete data among these different data types, and many researchers have developed a lot of investigations on incomplete data and put forward sufficient rough set models [4,21,38,53,62]. (2) *Extending binary relation*. There are two ways to extend the binary relation, one is to promote the number of binary relation in an information system, the other is to change the type of binary relation. For promoting the number of binary relation, we know that Pawlak rough set and its generalizations are constructed based on one equivalence relation in an information system, and the approximation space generated by one equivalence relation is considered as a granulation in an information system. In order to make rough set theory have a wider range of applications, Qian et al. extended Pawlak's single-granulation rough set to the multigranulation rough set model [41]. And later, many researchers have extended the multigranulation rough sets [19,24,25,33,42,43,47,49–51,60]. For changing the type of binary relation, one can use the similarity relation [29,48], tolerance relation [3,52,58], and dominance relation [3,11,12,16,20,27,28,53,61] to generalize the equivalence relation in Pawlak rough set model, so that relevant different rough set models based on different kinds of binary relations can be derived to meet different requirements.

In many real-life circumstances, the information system is no longer classical, namely, the binary relations in the information systems are not equivalence relations, but preference relations, such as dominance relation. We call this kind of information system as ordered information system [5,33,40,53]. It is vital to propose an extension called the dominance-based rough set approach (DRSA) to take into account the ordering properties of criteria [7,8,11]. The innovation is mainly based on substitution of the indiscernibility relation (equivalence relation) in an ordered information system by a dominance relation. Since Greco et al. initially studied DRSA in the year of 1998 [7,8], many scholars have investigated a variety of rough set models based on dominance relation to solve different problems [3,9,11,12,16,18,20,27,28,33,35,53,61]. Similar to Pawlak rough set model, the conditions imposed on the relationship between dominating set (dominated set) and upward union (downward union) in DRSA are so strict that there are no fault tolerance mechanisms [1,2,10,12,18,57]. Quantitative information about the degree of overlap of the dominating set (dominated set) and upward union (downward union) is not taken into account.

In fact, we could allow a certain degree of inconsistencies to exist in real-life applications. Let us take an example. In an ordered information system $(U, AT \cup d, V, f)$, for an object $x_i \in U$, the set of dominating x_i is denoted by $D_{R_A}(x_i)$. Assuming that the number of elements in $D_{R_A}(x_i)$ is very large, such as 10000, it means that the evaluations of these 10000 elements under the condition attribute set A are better than that of x_i, and in normal circumstances, the values of these 10000 elements under the decision attribute set d are also better than that of x_i. However, if the value of the decision attribute set of an element x_j in $D_{R_A}(x_i)$ is inferior than that of x_i for some reasons, this will lead to a very clear inconsistency situation. The purpose of DRSA is to construct a corresponding model to eliminate the inconsistencies between the decision attribute set and the condition attribute set in an ordered information system and to ensure that there is no inconsistency in the three decision regions obtained from the upper and lower approximations. In this case, however, the condition of defining DRSA is too strict to exclude x_i from the positive region because only one of the 10000 elements (namely 1/10000) causes the inconsistency. So it is proper to introduce the consistency level into the definition of upper and lower approximations to improve the flexibility of DRSA. Several works have been conducted regard to this aspect [1,10,12,13,18], however, there is still no relevant studies on introducing the consistency level from the double quantification viewpoint.

The purpose of this paper is to discuss the consistency decision analysis of rough set model with quantitative information in ordered information systems, and introduce a pair of single-quantitative variable consistency rough sets and two kinds of double-quantitative variable consistency rough sets to improve some results of DRSA. This is the motivation behind the research presented here. The paper is organized as follows. Related concepts and definitions in DRSA are reviewed briefly in Section 2. In Section 3, we introduce two kinds of quantitative consistency levels in an ordered information system from the relative accuracy and absolute accuracy viewpoint, respectively. In Section 4, the formation process of relative quantitative variable consistency dominance-based rough set approach (Rq-VC-DRSA) and absolute quantitative variable consistency dominance-based rough set approach (Aq-VC-DRSA) are presented. In Section 5, we discuss two types of double-quantitative variable consistency dominance-based rough set approach (Dq-VC-DRSA) models and the corresponding three-way decision rules. In Section 6, we develop an illustrative case study to do the consistency analysis of decision rules in DRSA and the proposed single-quantitative and double-quantitative variable consistency dominance-based rough set models by assuming a questionnaire survey of airline service quality. Finally, Section 7 covers some conclusions and further research directions.

2. Basic notions on DRSA

In this section, we review the basic concepts about DRSA in an ordered information system. The detailed description on DRSA could be referred to [7,8].

Definition 2.1. An information system is a tuple (U, AT, V, f), where U is a non-empty and finite set of objects, and $U = \{x_1, x_2, \cdots, x_n\}$; AT is a non-empty and finite set of attributes, and $AT = \{a_1, a_2, \cdots, a_m\}$; $f = \{f_l | U \to V_l, l \le m\}$, f_l is the value of a_l on $x \in U$, V_l is the domain of a_l, $a_l \in AT$. A decision information system is an information system $(U, AT \cup d, V, f)$, where $AT \cap d = \emptyset$, AT is the condition attribute set, while d is called the decision attribute set.

In an information system, if the domain of an attribute is ordered according to a decreasing or increasing preference, then the attribute is a criterion. An information system is called an ordered information system if all condition attributes are criteria. In an ordered information system, $\succcurlyeq_a$ is defined to denote the preference-ordered relation based on the condition attribute a. That is, $\forall a \in A$, if $x \succcurlyeq_a y$, then x dominates y in A, denoted by $xD_{R_A}y$; if $y \succcurlyeq_a x$, then x is dominated by y in A, denoted by $yD_{R_A}x$.

Definition 2.2. [7,8] Let (U, AT, V, f) be an ordered information system, $A \subseteq AT$. D_{R_A} is defined to be the dominance relation with respect to A as

$$D_{R_A} = \{(x, y) \in U \times U | f(x, a) \ge f(y, a), \forall a \in A\}.$$

$\forall x \in U$, two important sets of object x are obtained in the following:

(1) A set of objects dominating x, called A-dominating set, $D^+_{R_A}(x) = \{y \in U | yD_{R_A}x\}$;
(2) A set of objects dominated by x, called A-dominated set, $D^-_{R_A}(x) = \{y \in U | xD_{R_A}y\}$.

In an ordered information system with decision attribute set $(U, AT \cup d, V, f)$, the decision attribute set d makes a partition of U into a finite number of classes $\boldsymbol{Cl} = \{Cl_t, t \in T\}$ and $T = \{1, 2, \cdots, n\}$. Each $x \in U$ belongs to one and only one decision class Cl_t. To each decision attribute value v_{d_t}, $Cl_t = \{x \in U | f(x, d) = v_{d_t}\}$. The decision classes from $\boldsymbol{Cl}$ are preference-ordered according to increasing order of class indices, that is, for all $r, s \in T$ such that $r > s$, the objects from Cl_r are preferred to the objects from Cl_s. In other words, the classes $\boldsymbol{Cl}$ represent a comprehensive evaluation of the objects in U: the worst objects are in Cl_1, the best objects are in Cl_n, and the other objects belong to the remaining classes Cl_r, according to an evaluation improving with the index $r \in T$.

Due to the preference order in the set of classes $\boldsymbol{Cl}$, the sets to be approximated are not the particular classes but upward unions and downward unions of the classes, respectively.

$$Cl_t^{\succcurlyeq} = \bigcup_{s \ge t} Cl_s, \; Cl_t^{\preccurlyeq} = \bigcup_{s \le t} Cl_s, t = 1, 2, \cdots, n.$$

Definition 2.3. [7,8] Let (U, AT, V, f) be an ordered information system, $A \subseteq AT$, $t = 1, 2, \cdots, n$. The dominance-based rough set approach (DRSA) can be defined as

- the lower and upper approximations of the upward union $Cl_t^{\succcurlyeq}$ are

$$\underline{R_A}(Cl_t^{\succcurlyeq}) = \{x \in U : D_{R_A}^{+}(x) \subseteq Cl_t^{\succcurlyeq}\};$$
$$\overline{R_A}(Cl_t^{\succcurlyeq}) = \{x \in U : D_{R_A}^{-}(x) \cap Cl_t^{\succcurlyeq} \neq \emptyset\},$$

- the lower and upper approximations of the downward union $Cl_t^{\preccurlyeq}$ are

$$\underline{R_A}(Cl_t^{\preccurlyeq}) = \{x \in U : D_{R_A}^{-}(x) \subseteq Cl_t^{\preccurlyeq}\};$$
$$\overline{R_A}(Cl_t^{\preccurlyeq}) = \{x \in U : D_{R_A}^{+}(x) \cap Cl_t^{\preccurlyeq} \neq \emptyset\}.$$

For $t = 1, \cdots, n$, it is to verify that the upper approximations of $Cl_t^{\succcurlyeq}$ and $Cl_t^{\preccurlyeq}$ satisfy

$$\overline{R_A}(Cl_t^{\succcurlyeq}) = U - \underline{R_A}(U - Cl_t^{\succcurlyeq}),\ \overline{R_A}(Cl_t^{\preccurlyeq}) = U - \underline{R_A}(U - Cl_t^{\preccurlyeq}).$$

And the corresponding boundary regions of $Cl_t^{\succcurlyeq}$ and $Cl_t^{\preccurlyeq}$ are defined as

$$Bn(Cl_t^{\succcurlyeq}) = \overline{R_A}(Cl_t^{\succcurlyeq}) - \underline{R_A}(Cl_t^{\succcurlyeq}),\ Bn(Cl_t^{\preccurlyeq}) = \overline{R_A}(Cl_t^{\preccurlyeq}) - \underline{R_A}(Cl_t^{\preccurlyeq}).$$

The above definition of rough approximation is based on a strict application of the dominance principle. However, when defining non-ambiguous objects, it is reasonable to accept a limited proportion of negative examples, particularly for large data tables. Such an extended version of DRSA is called a variable consistency dominance-based rough set approach (VC-DRSA) [13]. The rough approximations defined within DRSA are based on consistency in the sense of dominance principle. It requires that objects having not-worse evaluation with respect to a set of considered criteria than a referent object cannot be assigned to a worse class than the referent object. However, some inconsistencies may decrease the cardinality of lower approximations to such an extent that it is impossible to discover strong patterns in the data, particularly when datasets are large. Thus, a relaxation of the strict dominance principle is worthwhile. The relaxation introduced in this paper to the DRSA model admits some inconsistent objects to the lower approximations, the range of this relaxation is controlled by an index called consistency level.

3. Quantitative consistency level

In the real world, we may come across the decision tables where the better condition attribute values are, the better the decision value is. This is the case where the condition attribute value of the object is consistent with the evaluation of the decision attribute of this object. But sometimes there is a situation that the condition attribute value of one object x is larger than that of another object y, but the evaluation of the decision attribute value of object x is worse than that of object y. This is the case when the condition attribute value of object is inconsistent with the evaluation of the decision attribute of that object. Then DRSA is proposed to deal with this inconsistency situation.

As we mentioned in the part of Introduction, the definition of lower and upper approximations in DRSA is too strict to deal with this inconsistency, and there is no fault-tolerance mechanism available. In fact, when the dataset is large, we could allow certain errors to exist. It is reasonable to introduce a certain degree of consistency threshold in the lower and upper approximations in DRSA. That is the reason why the VC-DRSA [13] was developed by Greco et al. In reference [13], authors introduced an indicator to define new lower and upper rough approximations, this indicator is actually the relative quantitative consistency level to be studied in this section.

When we consider introducing the consistency level into DRSA, we can construct the quantitative consistency level by the mentioned two kinds of quantification indexes, thus the relative and absolute quantitative consistency levels could be obtained.

Definition 3.1. For an attribute set $A \subseteq AT$, we say that $x \in U$ belongs to $Cl_t^{\succcurlyeq}$ with no ambiguity at relative quantitative consistency level $\alpha \in (0, 1]$, if $x \in Cl_t^{\succcurlyeq}$ and at least $\alpha * 100\%$ of all objects $y \in U$ dominating x with respect to A also belong to $Cl_t^{\succcurlyeq}$, i.e.

$$\frac{|D_{R_A}^{+}(x) \cap Cl_t^{\succcurlyeq}|}{|D_{R_A}^{+}(x)|} \geq \alpha.$$

The level α is called consistency level because it controls the degree of consistency between objects qualified as belonging to $Cl_t^{\succcurlyeq}$ without any ambiguity. In other words, if $\alpha < 1$, then $(1-\alpha) * 100\%$ of all objects $y \in U$ dominating x with respect to A do not belong to $Cl_t^{\succcurlyeq}$ and thus contradict the inclusion of x in $Cl_t^{\succcurlyeq}$.

Similarly, for attribute set A we say that $x \in U$ belongs to $Cl_t^{\preccurlyeq}$ with no ambiguity at relative quantitative consistency level $\alpha \in (0, 1]$, if $x \in Cl_t^{\preccurlyeq}$ and at least $\alpha * 100\%$ of all objects $y \in U$ dominated x with respect to A also belong to $Cl_t^{\preccurlyeq}$, i.e.

$$\frac{|D_{R_A}^-(x) \cap Cl_t^{\preccurlyeq}|}{|D_{R_A}^-(x)|} \geq \alpha.$$

Thus, for attribute set A, each object $x \in U$ is either ambiguous or non-ambiguous at consistency level α with respect to the upward union $Cl_t^{\succcurlyeq} (t = 2, 3, ..., n)$ or with respect to the downward union $Cl_t^{\preccurlyeq} (t = 1, 2, ..., n-1)$.

There are two kinds of absolute quantitative consistency levels, one is called internal absolute quantitative consistency level and the other is external absolute quantitative consistency level.

Definition 3.2. We say that $x \in U$ belongs to $Cl_t^{\succcurlyeq}$ with no ambiguity at internal absolute quantitative consistency level $k \in (0, |U|]$, if $x \in Cl_t^{\succcurlyeq}$ and the numbers of objects $y \in U$ dominating x with respect to A inside $Cl_t^{\succcurlyeq}$ exceed k, i.e.

$$|D_{R_A}^+(x) \cap Cl_t^{\succcurlyeq}| > k.$$

Similarly, $x \in U$ belongs to $Cl_t^{\preccurlyeq}$ with no ambiguity at internal absolute quantitative consistency level $k \in (0, |U|]$, if $x \in Cl_t^{\preccurlyeq}$ and the numbers of objects $y \in U$ dominated x with respect to A inside $Cl_t^{\preccurlyeq}$ exceed k, i.e.

$$|D_{R_A}^-(x) \cap Cl_t^{\preccurlyeq}| > k.$$

Definition 3.3. We say that $x \in U$ belongs to $Cl_t^{\succcurlyeq}$ with no ambiguity at external absolute quantitative consistency level $k \in (0, |U|]$, if $Cl_t^{\succcurlyeq}$ and the numbers of objects $y \in U$ dominating x with respect to A outside $Cl_t^{\succcurlyeq}$ are at most k, i.e.

$$|D_{R_A}^+(x)| - |D_{R_A}^+(x) \cap Cl_t^{\succcurlyeq}| \leq k.$$

Similarly, $x \in U$ belongs to $Cl_t^{\preccurlyeq}$ with no ambiguity at external absolute quantitative consistency level $k \in (0, |U|]$, if $Cl_t^{\preccurlyeq}$ and the numbers of objects $y \in U$ dominated x with respect to A outside $Cl_t^{\succcurlyeq}$ are at most k, i.e.

$$|D_{R_A}^-(x)| - |D_{R_A}^-(x) \cap Cl_t^{\preccurlyeq}| \leq k.$$

Based on the above two different forms of consistency level (relative quantitative and absolute quantitative), we can present the corresponding two single-quantitative variable consistency dominance-based rough set approaches, which will be studied in the next section.

4. Single-quantitative variable consistency dominance-based rough set approaches

In this section, we respectively investigate two kinds of single-quantitative variable consistency dominance-based rough set approach (Sq-VC-DRSA) models, which are relative quantitative variable consistency dominance-based rough set approach (Rq-VC-DRSA) and absolute quantitative variable consistency dominance-based rough set approach (Aq-VC-DRSA). Let us first discuss the following Rq-VC-DRSA.

4.1. Rq-VC-DRSA

Definition 4.1. Let $(U, AT \cup d, V, f)$ be an ordered information system, $A \subseteq AT$, $t = 1, 2, \cdots, n$, and the relative quantitative consistency level $\alpha \in (0, 1]$, the relative quantitative variable consistency dominance-based rough lower approximation of the upward union $Cl_t^{\succcurlyeq}$ with respect to attribute set A is defined as a set of objects $x \in Cl_t^{\succcurlyeq}$ whose relative quantitative consistency level are not less than α, denoted as

$$\underline{R_A}_\alpha(Cl_t^{\succcurlyeq}) = \{x \in Cl_t^{\succcurlyeq} : \frac{|D_{R_A}^+(x) \cap Cl_t^{\succcurlyeq}|}{|D_{R_A}^+(x)|} \geq \alpha\}.$$

By duality, the relative quantitative variable consistency dominance-based rough upper approximation of the upward union $Cl_t^{\succcurlyeq}$ can be defined as

$$\overline{R_A}_\alpha(Cl_t^{\succcurlyeq}) = U - \underline{R_A}_\alpha(U - Cl_t^{\succcurlyeq}).$$

Similarly, the relative quantitative variable consistency dominance-based rough lower and upper approximations of the downward union $Cl_t^{\preccurlyeq}$ are

$$\underline{R_A}_{\alpha}(Cl_t^{\preccurlyeq}) = \{x \in Cl_t^{\preccurlyeq} : \frac{|D_{R_A}^{-}(x) \cap Cl_t^{\preccurlyeq}|}{|D_{R_A}^{-}(x)|} \geq \alpha\};$$

$$\overline{R_A}_{\alpha}(Cl_t^{\preccurlyeq}) = U - \underline{R_A}_{\alpha}(U - Cl_t^{\preccurlyeq}).$$

Theorem 4.1. *The upper approximations of upward union $Cl_t^{\succcurlyeq}$ and downward union $Cl_t^{\preccurlyeq}$ in Rq-VC-DRSA have the following expressions.*

(1) $\overline{R_A}_{\alpha}(Cl_t^{\succcurlyeq}) = Cl_t^{\succcurlyeq} \cup \{x \in Cl_{t-1}^{\preccurlyeq} : \frac{|D_{R_A}^{-}(x) \cap Cl_t^{\succcurlyeq}|}{|D_{R_A}^{-}(x)|} > 1 - \alpha\};$

(2) $\overline{R_A}_{\alpha}(Cl_t^{\preccurlyeq}) = Cl_t^{\preccurlyeq} \cup \{x \in Cl_{t+1}^{\succcurlyeq} : \frac{|D_{R_A}^{+}(x) \cap Cl_t^{\preccurlyeq}|}{|D_{R_A}^{+}(x)|} > 1 - \alpha\}.$

Proof. (1) From Definition 4.1, we can derive the processes about the upper approximation of $Cl_t^{\succcurlyeq}$ in Rq-VC-DRSA as follows.

$$\begin{aligned}
\overline{R_A}_{\alpha}(Cl_t^{\succcurlyeq}) &= U - \underline{R_A}_{\alpha}(U - Cl_t^{\succcurlyeq}) = U - \underline{R_A}_{\alpha}(Cl_{t-1}^{\preccurlyeq}) \\
&= U - \{x \in Cl_{t-1}^{\preccurlyeq} : \frac{|D_{R_A}^{-}(x) \cap Cl_{t-1}^{\preccurlyeq}|}{|D_{R_A}^{-}(x)|} \geq \alpha\} \\
&= U - \{x \in Cl_{t-1}^{\preccurlyeq} : \frac{|D_{R_A}^{-}(x) \cap (U - Cl_t^{\succcurlyeq})|}{|D_{R_A}^{-}(x)|} \geq \alpha\} \\
&= U - \{x \in Cl_{t-1}^{\preccurlyeq} : \frac{|D_{R_A}^{-}(x) \cap Cl_t^{\succcurlyeq}|}{|D_{R_A}^{-}(x)|} \leq 1 - \alpha\} \\
&= Cl_t^{\succcurlyeq} \cup \{x \in Cl_{t-1}^{\preccurlyeq} : \frac{|D_{R_A}^{-}(x) \cap Cl_t^{\succcurlyeq}|}{|D_{R_A}^{-}(x)|} > 1 - \alpha\}.
\end{aligned}$$

(2) The processes about the upper approximation of $Cl_t^{\preccurlyeq}$ in Rq-VC-DRSA are derived as

$$\begin{aligned}
\overline{R_A}_{\alpha}(Cl_t^{\preccurlyeq}) &= U - \underline{R_A}_{\alpha}(U - Cl_t^{\preccurlyeq}) = U - \underline{R_A}_{\alpha}(Cl_{t+1}^{\succcurlyeq}) \\
&= U - \{x \in Cl_{t+1}^{\succcurlyeq} : \frac{|D_{R_A}^{+}(x) \cap Cl_{t+1}^{\succcurlyeq}|}{|D_{R_A}^{+}(x)|} \geq \alpha\} \\
&= U - \{x \in Cl_{t+1}^{\succcurlyeq} : \frac{|D_{R_A}^{+}(x) \cap (U - Cl_t^{\preccurlyeq})|}{|D_{R_A}^{+}(x)|} \geq \alpha\} \\
&= U - \{x \in Cl_{t+1}^{\succcurlyeq} : \frac{|D_{R_A}^{+}(x) \cap Cl_t^{\preccurlyeq}|}{|D_{R_A}^{+}(x)|} \leq 1 - \alpha\} \\
&= Cl_t^{\preccurlyeq} \cup \{x \in Cl_{t+1}^{\succcurlyeq} : \frac{|D_{R_A}^{+}(x) \cap Cl_t^{\preccurlyeq}|}{|D_{R_A}^{+}(x)|} > 1 - \alpha\}.
\end{aligned}$$

Then the proof process of the above theorem is completed. □

Theorem 4.2. *If $\alpha = 1$, then the Rq-VC-DRSA is degenerated into DRSA. In other words, Rq-VC-DRSA is a directional expansion of the DRSA.*

Proof. It is easy to verify the correctness of this theorem from Definition 4.1. □

All the objects belonging to $Cl_t^{\succcurlyeq}$ and $Cl_t^{\preccurlyeq}$ with some ambiguity at relative quantitative consistency level $\alpha \in (0, 1]$ constitute the boundary regions of $Cl_t^{\succcurlyeq}$ and $Cl_t^{\preccurlyeq}$.

For $x \in U$, we can define the positive region, negative region and boundary region of $Cl_t^{\succcurlyeq}$ as

$$Pos_\alpha(Cl_t^{\succcurlyeq}) = \underline{R_A}_\alpha(Cl_t^{\succcurlyeq}) = \{x \in Cl_t^{\succcurlyeq} : \frac{|D_{R_A}^+(x) \cap Cl_t^{\succcurlyeq}|}{|D_{R_A}^+(x)|} \geq \alpha\};$$

$$Neg_\alpha(Cl_t^{\succcurlyeq}) = U - \overline{R_A}_\alpha(Cl_t^{\succcurlyeq}) = \{x \in Cl_{t-1}^{\preccurlyeq} : \frac{|D_{R_A}^-(x) \cap Cl_{t-1}^{\preccurlyeq}|}{|D_{R_A}^-(x)|} \geq \alpha\};$$

$$Bn_\alpha(Cl_t^{\succcurlyeq}) = \overline{R_A}_\alpha(Cl_t^{\succcurlyeq}) - \underline{R_A}_\alpha(Cl_t^{\succcurlyeq}).$$

The corresponding three-way decision rules for the upward union $Cl_t^{\succcurlyeq}$ can be obtained:

(P) If $x \in Cl_t^{\succcurlyeq}$ and $\frac{|D_{R_A}^+(x) \cap Cl_t^{\succcurlyeq}|}{|D_{R_A}^+(x)|} \geq \alpha$, decide $Pos_\alpha(Cl_t^{\succcurlyeq})$;
(N) If $x \in U - Cl_t^{\succcurlyeq}$ and $\frac{|D_{R_A}^-(x) \cap Cl_{t-1}^{\preccurlyeq}|}{|D_{R_A}^-(x)|} \geq \alpha$, decide $Neg_\alpha(Cl_t^{\succcurlyeq})$;
(B) Otherwise, decide $Bn_\alpha(Cl_t^{\succcurlyeq})$.

For $x \in U$, the positive region, negative region and boundary region of $Cl_t^{\preccurlyeq}$ are defined as

$$Pos_\alpha(Cl_t^{\preccurlyeq}) = \underline{R_A}_\alpha(Cl_t^{\preccurlyeq}) = \{x \in Cl_t^{\preccurlyeq} : \frac{|D_{R_A}^-(x) \cap Cl_t^{\preccurlyeq}|}{|D_{R_A}^-(x)|} \geq \alpha\};$$

$$Neg_\alpha(Cl_t^{\preccurlyeq}) = U - \overline{R_A}_\alpha(Cl_t^{\preccurlyeq}) = \{x \in Cl_{t+1}^{\succcurlyeq} : \frac{|D_{R_A}^+(x) \cap Cl_{t+1}^{\succcurlyeq}|}{|D_{R_A}^+(x)|} \geq \alpha\};$$

$$Bn_\alpha(Cl_t^{\preccurlyeq}) = \overline{R_A}_\alpha(Cl_t^{\preccurlyeq}) - \underline{R_A}_\alpha(Cl_t^{\preccurlyeq}).$$

We can obtain the corresponding three-way decision rules for the downward union $Cl_t^{\preccurlyeq}$:

(P) If $x \in Cl_t^{\preccurlyeq}$ and $\frac{|D_{R_A}^-(x) \cap Cl_t^{\preccurlyeq}|}{|D_{R_A}^-(x)|} \geq \alpha$, decide $Pos_\alpha(Cl_t^{\preccurlyeq})$;
(N) If $x \in U - Cl_t^{\preccurlyeq}$ and $\frac{|D_{R_A}^+(x) \cap Cl_{t+1}^{\succcurlyeq}|}{|D_{R_A}^+(x)|} \geq \alpha$, decide $Neg_\alpha(Cl_t^{\preccurlyeq})$;
(B) Otherwise, decide $Bn_\alpha(Cl_t^{\preccurlyeq})$.

Theorem 4.3. *$\forall t \in T - \{1\}$ and $\forall A \subseteq AT$, $Bn_\alpha(Cl_t^{\succcurlyeq}) = Bn_\alpha(Cl_{t-1}^{\preccurlyeq})$.*

Proof. From the definition of the boundary region in Rq-VC-DRSA, we obtain that

$$\begin{aligned} Bn_\alpha(Cl_t^{\succcurlyeq}) &= \overline{R_A}_\alpha(Cl_t^{\succcurlyeq}) - \underline{R_A}_\alpha(Cl_t^{\succcurlyeq}) \\ &= \overline{R_A}_\alpha(U - Cl_{t-1}^{\preccurlyeq}) - \underline{R_A}_\alpha(U - Cl_{t-1}^{\preccurlyeq}) \\ &= U - \underline{R_A}_\alpha(Cl_{t-1}^{\preccurlyeq}) - (U - \overline{R_A}_\alpha(Cl_{t-1}^{\preccurlyeq})) \\ &= Bn_\alpha(Cl_{t-1}^{\preccurlyeq}). \end{aligned}$$

Then the proof is completed. □

Theorem 4.4. *Let $(U, AT \cup d, V, f)$ be an ordered information system, $A \subseteq AT$, $t \leq s \in \{1, 2, \cdots, n\}$, the lower and upper approximations in Rq-VC-DRSA satisfy the following properties.*
(1) $\underline{R_A}_\alpha(Cl_1^{\succcurlyeq}) = \overline{R_A}_\alpha(Cl_1^{\succcurlyeq}) = U$; $\underline{R_A}_\alpha(Cl_n^{\preccurlyeq}) = \overline{R_A}_\alpha(Cl_n^{\preccurlyeq}) = U$.
(2) $\underline{R_A}_\alpha(Cl_{n+1}^{\succcurlyeq}) = \overline{R_A}_\alpha(Cl_{n+1}^{\succcurlyeq}) = \emptyset$; $\underline{R_A}_\alpha(Cl_0^{\preccurlyeq}) = \overline{R_A}_\alpha(Cl_0^{\preccurlyeq}) = \emptyset$.
(3) $\underline{R_A}_\alpha(Cl_t^{\succcurlyeq}) \subseteq Cl_t^{\succcurlyeq} \subseteq \overline{R_A}_\alpha(Cl_t^{\succcurlyeq})$; $\underline{R_A}_\alpha(Cl_t^{\preccurlyeq}) \subseteq Cl_t^{\preccurlyeq} \subseteq \overline{R_A}_\alpha(Cl_t^{\preccurlyeq})$.
(4) $\underline{R_A}_\alpha(Cl_t^{\succcurlyeq}) \supseteq \underline{R_A}_\alpha(Cl_s^{\succcurlyeq})$, $\overline{R_A}_\alpha(Cl_t^{\succcurlyeq}) \supseteq \overline{R_A}_\alpha(Cl_s^{\succcurlyeq})$; $\underline{R_A}_\alpha(Cl_t^{\preccurlyeq}) \subseteq \underline{R_A}_\alpha(Cl_s^{\preccurlyeq})$, $\overline{R_A}_\alpha(Cl_t^{\preccurlyeq}) \subseteq \overline{R_A}_\alpha(Cl_s^{\preccurlyeq})$.

Proof. (1) As $Cl_1^{\succcurlyeq} = Cl_n^{\preccurlyeq} = U$, it is easy to obtain $\underline{R_A}_\alpha(Cl_1^{\succcurlyeq}) = \overline{R_A}_\alpha(Cl_1^{\succcurlyeq}) = U$ and $\underline{R_A}_\alpha(Cl_n^{\preccurlyeq}) = \overline{R_A}_\alpha(Cl_n^{\preccurlyeq}) = U$.

(2) As $Cl_{n+1}^{\succcurlyeq} = Cl_0^{\preccurlyeq} = \emptyset$, so $\underline{R_A}_\alpha(Cl_{n+1}^{\succcurlyeq}) = \overline{R_A}_\alpha(Cl_{n+1}^{\succcurlyeq}) = \emptyset$; $\underline{R_A}_\alpha(Cl_0^{\preccurlyeq}) = \overline{R_A}_\alpha(Cl_0^{\preccurlyeq}) = \emptyset$.

(3) For $Cl_t^{\succcurlyeq}$, from the Definition 4.1 and Theorem 4.1, we know that $\underline{R_A}_\alpha(Cl_t^{\succcurlyeq}) = \{x \in Cl_t^{\succcurlyeq} : \frac{|D_{R_A}^+(x) \cap Cl_t^{\succcurlyeq}|}{|D_{R_A}^+(x)|} \geq \alpha\}$ and $\overline{R_A}_\alpha(Cl_t^{\succcurlyeq}) = Cl_t^{\succcurlyeq} \cup \{x \in Cl_{t-1}^{\preccurlyeq} : \frac{|D_{R_A}^-(x) \cap Cl_t^{\succcurlyeq}|}{|D_{R_A}^-(x)|} > 1 - \alpha\}$, it is easy to see that $\forall x \in \underline{R_A}_\alpha(Cl_t^{\succcurlyeq})$, $x \in Cl_t^{\succcurlyeq}$; $\forall x \in Cl_t^{\succcurlyeq}$, $x \in \overline{R_A}_\alpha(Cl_t^{\succcurlyeq})$. The similar analysis for $Cl_t^{\preccurlyeq}$.

(4) $\forall x \in \underline{R_A}_\alpha(Cl_s^{\succcurlyeq})$, $x \in Cl_s^{\succcurlyeq}$ and $\frac{|D_{R_A}^+(x) \cap Cl_s^{\succcurlyeq}|}{|D_{R_A}^+(x)|} \geq \alpha$. As $t \leq s$, $Cl_t^{\succcurlyeq} \supseteq Cl_s^{\succcurlyeq}$, which means for $x \in Cl_s^{\succcurlyeq}$, we have $x \in Cl_t^{\succcurlyeq}$ and $\frac{|D_{R_A}^+(x) \cap Cl_t^{\succcurlyeq}|}{|D_{R_A}^+(x)|} \geq \alpha$, then $x \in \underline{R_A}_\alpha(Cl_t^{\succcurlyeq})$. So $\underline{R_A}_\alpha(Cl_t^{\succcurlyeq}) \supseteq \underline{R_A}_\alpha(Cl_s^{\succcurlyeq})$ is obtained. $\forall x \in \underline{R_A}_\alpha(Cl_t^{\preccurlyeq})$, $x \in Cl_t^{\preccurlyeq}$ and $\frac{|D_{R_A}^-(x) \cap Cl_t^{\preccurlyeq}|}{|D_{R_A}^-(x)|} \geq \alpha$. As $t \leq s$, $Cl_t^{\preccurlyeq} \subseteq Cl_s^{\preccurlyeq}$, which means, for $x \in Cl_t^{\preccurlyeq}$, we have $x \in Cl_s^{\preccurlyeq}$ and $\frac{|D_{R_A}^-(x) \cap Cl_s^{\preccurlyeq}|}{|D_{R_A}^-(x)|} \geq \alpha$, then $x \in \underline{R_A}_\alpha(Cl_s^{\preccurlyeq})$. So $\underline{R_A}_\alpha(Cl_t^{\preccurlyeq}) \subseteq \underline{R_A}_\alpha(Cl_s^{\preccurlyeq})$.

From the Definition 4.1, $\overline{R_A}_\alpha(Cl_t^{\succcurlyeq}) = U - \underline{R_A}_\alpha(Cl_{t-1}^{\preccurlyeq})$ and $\overline{R_A}_\alpha(Cl_s^{\succcurlyeq}) = U - \underline{R_A}_\alpha(Cl_{s-1}^{\preccurlyeq})$. According to the above proof process, $\underline{R_A}_\alpha(Cl_{t-1}^{\preccurlyeq}) \subseteq \underline{R_A}_\alpha(Cl_{s-1}^{\preccurlyeq})$, then $\overline{R_A}_\alpha(Cl_t^{\succcurlyeq}) \supseteq \overline{R_A}_\alpha(Cl_s^{\succcurlyeq})$. $\overline{R_A}_\alpha(Cl_t^{\preccurlyeq}) = U - \underline{R_A}_\alpha(Cl_{t-1}^{\succcurlyeq})$ and $\overline{R_A}_\alpha(Cl_s^{\preccurlyeq}) = U - \underline{R_A}_\alpha(Cl_{s-1}^{\succcurlyeq})$. $\underline{R_A}_\alpha(Cl_{t-1}^{\succcurlyeq}) \supseteq \underline{R_A}_\alpha(Cl_{s-1}^{\succcurlyeq})$, then $\overline{R_A}_\alpha(Cl_t^{\preccurlyeq}) \subseteq \overline{R_A}_\alpha(Cl_s^{\preccurlyeq})$.

Then the proof process of this theorem is completed. □

Introducing the same relative quantitative consistency level to each union in an ordered information system ignores the differences in size of unions to some extent. And the introduction of different relative quantitative consistency thresholds for each union ($\alpha_t^{\succcurlyeq}$ for $Cl_t^{\succcurlyeq}$, and $\alpha_t^{\preccurlyeq}$ for $Cl_t^{\preccurlyeq}$ instead of one α for all unions) may solve the problem with different size of unions. However, for $t \leq s \in \{1, 2, \cdots, n\}$, $\alpha_t^{\succcurlyeq} \neq \alpha_s^{\succcurlyeq}$, $\alpha_{t-1}^{\preccurlyeq} \neq \alpha_{s-1}^{\preccurlyeq}$, $\alpha_t^{\preccurlyeq} \neq \alpha_s^{\preccurlyeq}$ and $\alpha_{t+1}^{\succcurlyeq} \neq \alpha_{s+1}^{\succcurlyeq}$ generally hold with different relative quantitative consistency thresholds for each union. Then it is hard to compare the inclusion relationship between $\underline{R_A}_\alpha(Cl_t^{\succcurlyeq})$ and $\underline{R_A}_\alpha(Cl_s^{\succcurlyeq})$, $\underline{R_A}_\alpha(Cl_t^{\preccurlyeq})$ and $\underline{R_A}_\alpha(Cl_s^{\preccurlyeq})$, $\overline{R_A}_\alpha(Cl_t^{\succcurlyeq})$ and $\overline{R_A}_\alpha(Cl_s^{\succcurlyeq})$, $\overline{R_A}_\alpha(Cl_t^{\preccurlyeq})$ and $\overline{R_A}_\alpha(Cl_s^{\preccurlyeq})$, which can be shown in item (4) of Theorem 4.4. The same issue for the Aq-VC-DRSA and Dq-VC-RDSA to be studied in the Subsection 4.2 and Section 5.

After discussing Rq-VC-DRSA, we will investigate the Aq-VC-DRSA in the following subsection. Different from Rq-VC-DRSA, the Aq-VC-DRSA uses absolute quantitative consistency level to define the corresponding rough approximations.

4.2. Aq-VC-DRSA

Given an absolute quantitative consistency level $0 < k \leq |U|$, the absolute quantitative variable consistency dominance-based rough lower approximation of upward union $Cl_t^{\succcurlyeq}$ with respect to attribute set A is defined as a set of objects $x \in Cl_t^{\succcurlyeq}$ whose external absolute quantitative consistency level are not more than k. Let us focus on the following Definition 4.2.

Definition 4.2. Let $(U, AT \cup d, V, f)$ be an ordered information system, $A \subseteq AT$, $t = 1, 2, \cdots, n$. Suppose k is a non-negative integer called "grade". The absolute quantitative variable consistency dominance-based rough lower and upper approximations of the upward union $Cl_t^{\succcurlyeq}$ are

$$\underline{R_A}_k(Cl_t^{\succcurlyeq}) = \{x \in Cl_t^{\succcurlyeq} : |D_{R_A}^+(x)| - |D_{R_A}^+(x) \cap Cl_t^{\succcurlyeq}| \leq k\};$$
$$\overline{R_A}_k(Cl_t^{\succcurlyeq}) = U - \underline{R_A}_k(U - Cl_t^{\succcurlyeq}).$$

Similarly, the absolute quantitative variable consistency dominance-based rough lower and upper approximations of the downward union $Cl_t^{\preccurlyeq}$ are

$$\underline{R_A}_k(Cl_t^{\preccurlyeq}) = \{x \in Cl_t^{\preccurlyeq} : |D_{R_A}^-(x)| - |D_{R_A}^-(x) \cap Cl_t^{\preccurlyeq}| \leq k\};$$
$$\overline{R_A}_k(Cl_t^{\preccurlyeq}) = U - \underline{R_A}_k(U - Cl_t^{\preccurlyeq}).$$

Theorem 4.5. *The upper approximations of upward union $Cl_t^{\succcurlyeq}$ and downward union $Cl_t^{\preccurlyeq}$ in Aq-VC-DRSA have the following expressions.*

(1) $\overline{R_A}_k(Cl_t^{\succcurlyeq}) = Cl_t^{\succcurlyeq} \cup \{x \in Cl_{t-1}^{\preccurlyeq} : |D_{R_A}^-(x) \cap Cl_t^{\succcurlyeq}| > k\}$;
(2) $\overline{R_A}_k(Cl_t^{\preccurlyeq}) = Cl_t^{\preccurlyeq} \cup \{x \in Cl_{t+1}^{\succcurlyeq} : |D_{R_A}^+(x) \cap Cl_t^{\preccurlyeq}| > k\}$.

Proof. (1) From Definition 4.2, we can derive the processes about the upper approximation of $Cl_t^{\succcurlyeq}$ in Aq-VC-DRSA as follows.

$$\begin{aligned}\overline{R_A}_k(Cl_t^{\succcurlyeq}) &= U - \underline{R_A}_k(U - Cl_t^{\succcurlyeq}) = U - \underline{R_A}_k(Cl_{t-1}^{\preccurlyeq})\\ &= U - \{x \in Cl_{t-1}^{\preccurlyeq} : |D_{R_A}^{-}(x)| - |D_{R_A}^{-}(x) \cap Cl_{t-1}^{\preccurlyeq}| \le k\}\\ &= U - \{x \in Cl_{t-1}^{\preccurlyeq} : |D_{R_A}^{-}(x) - D_{R_A}^{-}(x) \cap Cl_{t-1}^{\preccurlyeq}| \le k\}\\ &= U - \{x \in Cl_{t-1}^{\preccurlyeq} : |D_{R_A}^{-}(x) \cap Cl_t^{\succcurlyeq}| \le k\}\\ &= Cl_t^{\succcurlyeq} \cup \{x \in Cl_{t-1}^{\preccurlyeq} : |D_{R_A}^{-}(x) \cap Cl_t^{\succcurlyeq}| > k\}\end{aligned}$$

(2) The processes about the upper approximation of $Cl_t^{\preccurlyeq}$ in Aq-VC-DRSA are derived as

$$\begin{aligned}\overline{R_A}_k(Cl_t^{\preccurlyeq}) &= U - \underline{R_A}_k(U - Cl_t^{\preccurlyeq}) = U - \underline{R_A}_k(Cl_{t+1}^{\succcurlyeq})\\ &= U - \{x \in Cl_{t+1}^{\succcurlyeq} : |D_{R_A}^{+}(x)| - |D_{R_A}^{+}(x) \cap Cl_{t+1}^{\succcurlyeq}| \le k\}\\ &= U - \{x \in Cl_{t+1}^{\succcurlyeq} : |D_{R_A}^{+}(x) - D_{R_A}^{+}(x) \cap Cl_{t+1}^{\succcurlyeq}| \le k\}\\ &= U - \{x \in Cl_{t+1}^{\succcurlyeq} : |D_{R_A}^{+}(x) \cap Cl_t^{\preccurlyeq}| \le k\}\\ &= Cl_t^{\preccurlyeq} \cup \{x \in Cl_{t+1}^{\succcurlyeq} : |D_{R_A}^{+}(x) \cap Cl_t^{\preccurlyeq}| > k\}\end{aligned}$$

Then the proof process of the above theorem is completed. □

Theorem 4.6. *If $k = 0$, then the Aq-VC-DRSA is degenerated into DRSA. In other words, Aq-VC-DRSA is a directional expansion of the DRSA.*

Proof. It is easy to verify the correctness of this theorem from Definition 4.2. □

All the objects belonging to $Cl_t^{\succcurlyeq}$ and $Cl_t^{\preccurlyeq}$ with some ambiguity at absolute quantitative consistency level $k \in (0, |U|]$ constitute the boundary regions of $Cl_t^{\succcurlyeq}$ and $Cl_t^{\preccurlyeq}$.

For $x \in U$, we can define the positive region, negative region and boundary region of $Cl_t^{\succcurlyeq}$ as

$$\begin{aligned}Pos_k(Cl_t^{\succcurlyeq}) &= \underline{R_A}_k(Cl_t^{\succcurlyeq})\\ &= \{x \in Cl_t^{\succcurlyeq} : |D_{R_A}^{+}(x)| - |D_{R_A}^{+}(x) \cap Cl_t^{\succcurlyeq}| \le k\};\\ Neg_k(Cl_t^{\succcurlyeq}) &= U - \overline{R_A}_k(Cl_t^{\succcurlyeq})\\ &= \{x \in Cl_{t-1}^{\preccurlyeq} : |D_{R_A}^{-}(x)| - |D_{R_A}^{-}(x) \cap Cl_{t-1}^{\preccurlyeq}| \le k\};\\ Bn_k(Cl_t^{\succcurlyeq}) &= \overline{R_A}_k(Cl_t^{\succcurlyeq}) - \underline{R_A}_k(Cl_t^{\succcurlyeq}).\end{aligned}$$

The corresponding three-way decision rules for the upward union $Cl_t^{\succcurlyeq}$ can be obtained:

(P) If $x \in Cl_t^{\succcurlyeq}$ and $|D_{R_A}^{+}(x)| - |D_{R_A}^{+}(x) \cap Cl_t^{\succcurlyeq}| \le k$, decide $Pos_k(Cl_t^{\succcurlyeq})$;
(N) If $x \in U - Cl_t^{\succcurlyeq}$ and $|D_{R_A}^{-}(x)| - |D_{R_A}^{-}(x) \cap Cl_{t-1}^{\preccurlyeq}| \le k$, decide $Neg_k(Cl_t^{\succcurlyeq})$;
(B) Otherwise, decide $Bn_k(Cl_t^{\succcurlyeq})$.

For $x \in U$, the positive region, negative region and boundary region of $Cl_t^{\preccurlyeq}$ are defined as

$$\begin{aligned}Pos_k(Cl_t^{\preccurlyeq}) &= \underline{R_A}_k(Cl_t^{\preccurlyeq})\\ &= \{x \in Cl_t^{\preccurlyeq} : |D_{R_A}^{-}(x)| - |D_{R_A}^{-}(x) \cap Cl_t^{\preccurlyeq}| \le k\};\\ Neg_k(Cl_t^{\preccurlyeq}) &= U - \overline{R_A}_k(Cl_t^{\preccurlyeq})\\ &= \{x \in Cl_{t+1}^{\succcurlyeq} : |D_{R_A}^{+}(x)| - |D_{R_A}^{+}(x) \cap Cl_{t+1}^{\succcurlyeq}| \le k\};\\ Bn_k(Cl_t^{\preccurlyeq}) &= \overline{R_A}_k(Cl_t^{\preccurlyeq}) - \underline{R_A}_k(Cl_t^{\preccurlyeq}).\end{aligned}$$

We can obtain the corresponding three-way decision rules for the downward union $Cl_t^{\preccurlyeq}$:

(P) If $x \in Cl_t^{\preccurlyeq}$ and $|D_{R_A}^{-}(x)| - |D_{R_A}^{-}(x) \cap Cl_t^{\preccurlyeq}| \le k$, decide $Pos_k(Cl_t^{\preccurlyeq})$;
(N) If $x \in U - Cl_t^{\preccurlyeq}$ and $|D_{R_A}^{+}(x)| - |D_{R_A}^{+}(x) \cap Cl_{t+1}^{\succcurlyeq}| \le k$, decide $Neg_k(Cl_t^{\preccurlyeq})$;
(B) Otherwise, decide $Bn_k(Cl_t^{\preccurlyeq})$.

Theorem 4.7. *$\forall t \in T - \{1\}$ and $\forall A \subseteq AT$, $Bn_k(Cl_t^{\succcurlyeq}) = Bn_k(Cl_{t-1}^{\preccurlyeq})$.*

Proof. From the definition of the boundary region in Aq-VC-DRSA, we can obtain that

$$\begin{aligned} Bn_k(Cl_t^{\succcurlyeq}) &= \overline{R_A}_k(Cl_t^{\succcurlyeq}) - \underline{R_{A_k}}(Cl_t^{\succcurlyeq}) \\ &= \overline{R_A}_k(U - Cl_{t-1}^{\preccurlyeq}) - \underline{R_{A_k}}(U - Cl_{t-1}^{\preccurlyeq}) \\ &= U - \underline{R_{A_k}}(Cl_{t-1}^{\preccurlyeq}) - (U - \overline{R_A}_k(Cl_{t-1}^{\preccurlyeq})) \\ &= \overline{R_A}_k(Cl_{t-1}^{\preccurlyeq}) - \underline{R_{A_k}}(Cl_{t-1}^{\preccurlyeq}) \\ &= Bn_k(Cl_{t-1}^{\preccurlyeq}). \end{aligned}$$

Then the proof is completed. □

Theorem 4.8. *Let $(U, AT \cup d, V, f)$ be an ordered information system, $A \subseteq AT$, $t = 1, 2, \cdots, n$, the lower and upper approximations in Aq-VC-DRSA satisfy*

(1) $\underline{R_{A_k}}(Cl_1^{\succcurlyeq}) = \overline{R_A}_k(Cl_1^{\succcurlyeq}) = U$; $\underline{R_{A_k}}(Cl_n^{\preccurlyeq}) = \overline{R_A}_k(Cl_n^{\preccurlyeq}) = U$.
(2) $\underline{R_{A_k}}(Cl_{n+1}^{\succcurlyeq}) = \overline{R_A}_k(Cl_{n+1}^{\succcurlyeq}) = \emptyset$; $\underline{R_{A_k}}(Cl_0^{\preccurlyeq}) = \overline{R_A}_k(Cl_0^{\preccurlyeq}) = \emptyset$.
(3) $\underline{R_{A_k}}(Cl_t^{\succcurlyeq}) \subseteq Cl_t^{\succcurlyeq} \subseteq \overline{R_A}_k(Cl_t^{\succcurlyeq})$; $\underline{R_{A_k}}(Cl_t^{\preccurlyeq}) \subseteq Cl_t^{\preccurlyeq} \subseteq \overline{R_A}_k(Cl_t^{\preccurlyeq})$.
(4) $\underline{R_{A_k}}(Cl_t^{\succcurlyeq}) \supseteq \underline{R_{A_k}}(Cl_s^{\succcurlyeq})$, $\overline{R_A}_k(Cl_t^{\succcurlyeq}) \supseteq \overline{R_A}_k(Cl_s^{\succcurlyeq})$; $\underline{R_{A_k}}(Cl_t^{\preccurlyeq}) \subseteq \underline{R_{A_k}}(Cl_s^{\preccurlyeq})$, $\overline{R_A}_k(Cl_t^{\preccurlyeq}) \subseteq \overline{R_A}_k(Cl_s^{\preccurlyeq})$.

Proof. (1) As $Cl_1^{\succcurlyeq} = Cl_n^{\preccurlyeq} = U$, it is easy to obtain $\underline{R_{A_k}}(Cl_1^{\succcurlyeq}) = \overline{R_A}_k(Cl_1^{\succcurlyeq}) = U$ and $\underline{R_{A_k}}(Cl_n^{\preccurlyeq}) = \overline{R_A}_k(Cl_n^{\preccurlyeq}) = U$.

(2) As $Cl_{n+1}^{\succcurlyeq} = Cl_0^{\preccurlyeq} = \emptyset$, so $\underline{R_{A_k}}(Cl_{n+1}^{\succcurlyeq}) = \overline{R_A}_k(Cl_{n+1}^{\succcurlyeq}) = \emptyset$; $\underline{R_{A_k}}(Cl_0^{\preccurlyeq}) = \overline{R_A}_k(Cl_0^{\preccurlyeq}) = \emptyset$.

(3) For $Cl_t^{\succcurlyeq}$, from the Definition 4.2 and Theorem 4.5, we know that $\underline{R_{A_k}}(Cl_t^{\succcurlyeq}) = \{x \in Cl_t^{\succcurlyeq} : |D_{R_A}^{+}(x)| - |D_{R_A}^{+}(x) \cap Cl_t^{\succcurlyeq}| \le k\}$ and $\overline{R_A}_k(Cl_t^{\succcurlyeq}) = Cl_t^{\succcurlyeq} \cup \{x \in Cl_{t-1}^{\preccurlyeq} : |D_{R_A}^{-}(x) \cap Cl_t^{\succcurlyeq}| > k\}$, it is easy to see that $\forall x \in \underline{R_{A_k}}(Cl_t^{\succcurlyeq})$, $x \in Cl_t^{\succcurlyeq}$; $\forall x \in Cl_t^{\succcurlyeq}$, $x \in \overline{R_A}_k(Cl_t^{\succcurlyeq})$. The similar analysis for $Cl_t^{\preccurlyeq}$.

(4) $\forall x \in \underline{R_{A_k}}(Cl_s^{\succcurlyeq})$, $x \in Cl_s^{\succcurlyeq}$ and $|D_{R_A}^{+}(x)| - |D_{R_A}^{+}(x) \cap Cl_s^{\succcurlyeq}| \le k$. As $t \le s$, $Cl_t^{\succcurlyeq} \supseteq Cl_s^{\succcurlyeq}$, which means for $x \in Cl_s^{\succcurlyeq}$, we have $x \in Cl_t^{\succcurlyeq}$ and $|D_{R_A}^{+}(x)| - |D_{R_A}^{+}(x) \cap Cl_t^{\succcurlyeq}| \le k$, then $x \in \underline{R_{A_k}}(Cl_t^{\succcurlyeq})$. So $\underline{R_{A_k}}(Cl_t^{\succcurlyeq}) \supseteq \underline{R_{A_k}}(Cl_s^{\succcurlyeq})$ is obtained. $\forall x \in \underline{R_{A_k}}(Cl_t^{\preccurlyeq})$, $x \in Cl_t^{\preccurlyeq}$ and $|D_{R_A}^{-}(x)| - |D_{R_A}^{-}(x) \cap Cl_t^{\preccurlyeq}| \le k$. As $t \le s$, $Cl_t^{\preccurlyeq} \subseteq Cl_s^{\preccurlyeq}$, which means for $x \in Cl_t^{\preccurlyeq}$, we have $x \in Cl_s^{\preccurlyeq}$ and $|D_{R_A}^{-}(x)| - |D_{R_A}^{-}(x) \cap Cl_s^{\preccurlyeq}| \le k$, then $x \in \underline{R_{A_\alpha}}(Cl_s^{\preccurlyeq})$. So $\underline{R_{A_k}}(Cl_t^{\preccurlyeq}) \subseteq \underline{R_{A_k}}(Cl_s^{\preccurlyeq})$.

From the Definition 4.2, $\overline{R_A}_k(Cl_t^{\succcurlyeq}) = U - \underline{R_{A_k}}(Cl_{t-1}^{\preccurlyeq})$ and $\overline{R_A}_k(Cl_s^{\succcurlyeq}) = U - \underline{R_{A_k}}(Cl_{s-1}^{\preccurlyeq})$. According to the above proof process, $\underline{R_{A_k}}(Cl_{t-1}^{\preccurlyeq}) \subseteq \underline{R_{A_k}}(Cl_{s-1}^{\preccurlyeq})$, then $\overline{R_A}_k(Cl_t^{\succcurlyeq}) \supseteq \overline{R_A}_k(Cl_s^{\succcurlyeq})$. $\overline{R_A}_k(Cl_t^{\preccurlyeq}) = U - \underline{R_{A_k}}(Cl_{t-1}^{\succcurlyeq})$ and $\overline{R_A}_k(Cl_s^{\preccurlyeq}) = U - \underline{R_{A_k}}(Cl_{s-1}^{\succcurlyeq})$. $\underline{R_{A_k}}(Cl_{t-1}^{\succcurlyeq}) \supseteq \underline{R_{A_k}}(Cl_{s-1}^{\succcurlyeq})$, then $\overline{R_A}_k(Cl_t^{\preccurlyeq}) \subseteq \overline{R_A}_k(Cl_s^{\preccurlyeq})$.

Then the proof process of this theorem is completed. □

The Rq-VC-DRSA utilizes relative quantitative consistency measure to calculate the approximations, there is no information regarding the absolute quantitative consistency level in Rq-VC-DRSA; The Aq-VC-DRSA utilizes absolute quantitative consistency measure to obtain the approximations, there is no information regarding the relative quantitative consistency level in Aq-VC-DRSA. Relative and absolute quantitative information are two kinds of quantification methodology in certain applications. Let us take the dominating set $D_{R_A}^{+}(x)$ and the upward union $Cl_t^{\succcurlyeq}$ for example. For two dominating sets $|D_{R_A}^{+}(x_1)| \ne |D_{R_A}^{+}(x_2)|$, if $\frac{|D_{R_A}^{+}(x_1) \cap Cl_t^{\succcurlyeq}|}{|D_{R_A}^{+}(x_1)|} = \frac{|D_{R_A}^{+}(x_2) \cap Cl_t^{\succcurlyeq}|}{|D_{R_A}^{+}(x_2)|}$, then x_1 and x_2 are indiscernible or equal in Rq-VC-DRSA. However, $|D_{R_A}^{+}(x_1)| \ne |D_{R_A}^{+}(x_2)|$ or $|D_{R_A}^{+}(x_1) \cap Cl_t^{\succcurlyeq}| \ne |D_{R_A}^{+}(x_2) \cap Cl_t^{\succcurlyeq}|$, and x_1 and x_2 can be discerned by introducing the absolute quantitative information $|D_{R_A}^{+}(x)|$ or $|D_{R_A}^{+}(x) \cap Cl_t^{\succcurlyeq}|$. The double quantification formed by adding the absolute quantitative information can improve the descriptive abilities of Rq-VC-DRSA and expand the range of applicability. The same analysis for adding relative quantitative information to Aq-VC-DRSA. For two dominating sets $|D_{R_A}^{+}(x_1)| \ne |D_{R_A}^{+}(x_2)|$, if $|D_{R_A}^{+}(x_1)| - |D_{R_A}^{+}(x_1) \cap Cl_t^{\succcurlyeq}| = |D_{R_A}^{+}(x_2)| - |D_{R_A}^{+}(x_2) \cap Cl_t^{\succcurlyeq}|$, then x_1 and x_2 are indiscernible or equal in Aq-VC-DRSA. However, their relative quantitative information may satisfy $\frac{|D_{R_A}^{+}(x_1) \cap Cl_t^{\succcurlyeq}|}{|D_{R_A}^{+}(x_1)|} \ne \frac{|D_{R_A}^{+}(x_2) \cap Cl_t^{\succcurlyeq}|}{|D_{R_A}^{+}(x_2)|}$, in this case, x_1 and x_2 can be discerned by introducing the relative quantitative information $\frac{|D_{R_A}^{+}(x) \cap Cl_t^{\succcurlyeq}|}{|D_{R_A}^{+}(x)|}$. Thus, it is necessary to implement the double quantification using the relative and absolute quantitative consistency levels simultaneously in the DRSA.

5. Double-quantitative variable consistency dominance-based rough set approaches

In the previous section, we present the single-quantitative dominance-based rough approximations based on two different types of quantitative consistency thresholds. In what follows, we investigate two kinds of Dq-VC-DRSA (double-quantitative variable consistency dominance-based rough set approach) models. These two models have their own specific application background, and we should decide which model to use according to the actual application requirements. Let us start with the first kind of Dq-VC-DRSA, denoted as DqI-VC-DRSA.

5.1. DqI-VC-DRSA

If the lower approximation must contain two kinds of quantitative consistency levels, then the DqI-VC-DRSA in the following Definition 5.1 can be applied.

Definition 5.1. Let $(U, AT \cup d, V, f)$ be an ordered information system, $A \subseteq AT$, $t = 1, 2, \cdots, n$. The DqI-VC-DRSA can be defined as follows.

- The first kind of double-quantitative variable consistency dominance-based rough approximations of the upward union $Cl_t^{\succcurlyeq}$ are defined as

$$\begin{aligned}\underline{R_A}^{I}_{(\alpha,k)}(Cl_t^{\succcurlyeq}) =&\{x \in Cl_t^{\succcurlyeq} : \frac{|D_{R_A}^{+}(x) \cap Cl_t^{\succcurlyeq}|}{|D_{R_A}^{+}(x)|} \geq \alpha \\ &\wedge |D_{R_A}^{+}(x)| - |D_{R_A}^{+}(x) \cap Cl_t^{\succcurlyeq}| \leq k\};\\ \overline{R_A}^{I}_{(\alpha,k)}(Cl_t^{\succcurlyeq}) =& U - \underline{R_A}^{I}_{(\alpha,k)}(U - Cl_t^{\succcurlyeq}).\end{aligned}$$

- The first kind of double-quantitative variable consistency dominance-based rough approximations of the downward union $Cl_t^{\preccurlyeq}$ are

$$\begin{aligned}\underline{R_A}^{I}_{(\alpha,k)}(Cl_t^{\preccurlyeq}) =&\{x \in Cl_t^{\preccurlyeq} : \frac{|D_{R_A}^{-}(x) \cap Cl_t^{\preccurlyeq}|}{|D_{R_A}^{-}(x)|} \geq \alpha \\ &\wedge |D_{R_A}^{-}(x)| - |D_{R_A}^{-}(x) \cap Cl_t^{\preccurlyeq}| \leq k\};\\ \overline{R_A}^{I}_{(\alpha,k)}(Cl_t^{\preccurlyeq}) =& U - \underline{R_A}^{I}_{(\alpha,k)}(U - Cl_t^{\preccurlyeq}).\end{aligned}$$

Theorem 5.1. *The upper approximations of upward union $Cl_t^{\succcurlyeq}$ and downward union $Cl_t^{\preccurlyeq}$ in DqI-VC-DRSA have the following expressions.*

(1) $\overline{R_A}^{I}_{(\alpha,k)}(Cl_t^{\succcurlyeq}) = Cl_t^{\succcurlyeq} \cup \{x \in Cl_{t-1}^{\preccurlyeq} : \frac{|D_{R_A}^{-}(x) \cap Cl_t^{\succcurlyeq}|}{|D_{R_A}^{-}(x)|} > 1 - \alpha \vee |D_{R_A}^{-}(x) \cap Cl_t^{\succcurlyeq}| > k\}$;

(2) $\overline{R_A}^{I}_{(\alpha,k)}(Cl_t^{\preccurlyeq}) = Cl_t^{\preccurlyeq} \cup \{x \in Cl_{t+1}^{\succcurlyeq} : \frac{|D_{R_A}^{+}(x) \cap Cl_t^{\preccurlyeq}|}{|D_{R_A}^{+}(x)|} > 1 - \alpha \vee |D_{R_A}^{+}(x) \cap Cl_t^{\preccurlyeq}| > k\}$.

Proof. (1) From Definition 5.1, we can derive the processes about the upper approximation of $Cl_t^{\succcurlyeq}$ in DqI-VC-DRSA as follows.

$$\begin{aligned}\overline{R_A}^{I}_{(\alpha,k)}(Cl_t^{\succcurlyeq}) &= U - \underline{R_A}^{I}_{(\alpha,k)}(U - Cl_t^{\succcurlyeq}) = U - \underline{R_A}^{I}_{(\alpha,k)}(Cl_{t-1}^{\preccurlyeq})\\ &= U - \{x \in Cl_{t-1}^{\preccurlyeq} : \frac{|D_{R_A}^{-}(x) \cap Cl_{t-1}^{\preccurlyeq}|}{|D_{R_A}^{-}(x)|} \geq \alpha \wedge |D_{R_A}^{-}(x)| - |D_{R_A}^{-}(x) \cap Cl_{t-1}^{\preccurlyeq}| \leq k\}\\ &= U - \{x \in Cl_{t-1}^{\preccurlyeq} : \frac{|D_{R_A}^{-}(x) \cap Cl_t^{\succcurlyeq}|}{|D_{R_A}^{-}(x)|} \leq 1 - \alpha \wedge |D_{R_A}^{-}(x) - D_{R_A}^{-}(x) \cap Cl_{t-1}^{\preccurlyeq}| \leq k\}\\ &= U - \{x \in Cl_{t-1}^{\preccurlyeq} : \frac{|D_{R_A}^{-}(x) \cap Cl_t^{\succcurlyeq}|}{|D_{R_A}^{-}(x)|} \leq 1 - \alpha\} \cap \{x \in Cl_{t-1}^{\preccurlyeq} : |D_{R_A}^{-}(x) - D_{R_A}^{-}(x) \cap Cl_{t-1}^{\preccurlyeq}| \leq k\}\\ &= (U - \{x \in Cl_{t-1}^{\preccurlyeq} : \frac{|D_{R_A}^{-}(x) \cap Cl_t^{\succcurlyeq}|}{|D_{R_A}^{-}(x)|} \leq 1 - \alpha\}) \cup (U - \{x \in Cl_{t-1}^{\preccurlyeq} : |D_{R_A}^{-}(x) \cap Cl_t^{\succcurlyeq}| \leq k\})\end{aligned}$$

$$= (Cl_t^{\succeq} \cup \{x \in Cl_{t-1}^{\preceq} : \frac{|D_{R_A}^{-}(x) \cap Cl_t^{\succeq}|}{|D_{R_A}^{-}(x)|} > 1-\alpha\}) \cup (Cl_t^{\succeq} \cup \{x \in Cl_{t-1}^{\preceq} : |D_{R_A}^{-}(x) \cap Cl_t^{\succeq}| > k\})$$

$$= Cl_t^{\succeq} \cup \{x \in Cl_{t-1}^{\preceq} : \frac{|D_{R_A}^{-}(x) \cap Cl_t^{\succeq}|}{|D_{R_A}^{-}(x)|} > 1-\alpha \vee |D_{R_A}^{-}(x) \cap Cl_t^{\succeq}| > k\}$$

(2) The processes about the upper approximation of $Cl_t^{\preceq}$ in DqI-VC-DRSA are derived as

$$\overline{RA}^{I}_{(\alpha,k)}(Cl_t^{\preceq}) = U - \underline{RA}^{I}_{(\alpha,k)}(U - Cl_t^{\preceq}) = U - \underline{RA}^{I}_{(\alpha,k)}(Cl_{t+1}^{\succeq})$$

$$= U - \{x \in Cl_{t+1}^{\succeq} : \frac{|D_{R_A}^{+}(x) \cap Cl_{t+1}^{\succeq}|}{|D_{R_A}^{+}(x)|} \geq \alpha \wedge |D_{R_A}^{+}(x)| - |D_{R_A}^{+}(x) \cap Cl_{t+1}^{\succeq}| \leq k\}$$

$$= U - \{x \in Cl_{t+1}^{\succeq} : \frac{|D_{R_A}^{+}(x) \cap (U - Cl_t^{\preceq})|}{|D_{R_A}^{+}(x)|} \geq \alpha \wedge |D_{R_A}^{+}(x) - D_{R_A}^{+}(x) \cap Cl_{t+1}^{\succeq}| \leq k\}$$

$$= U - \{x \in Cl_{t+1}^{\succeq} : \frac{|D_{R_A}^{+}(x) \cap Cl_t^{\preceq}|}{|D_{R_A}^{+}(x)|} \leq 1-\alpha\} \cap \{x \in Cl_{t+1}^{\succeq} : |D_{R_A}^{+}(x) \cap Cl_t^{\preceq}| \leq k\}$$

$$= (U - \{x \in Cl_{t+1}^{\preceq} : \frac{|D_{R_A}^{+}(x) \cap Cl_t^{\succeq}|}{|D_{R_A}^{+}(x)|} \leq 1-\alpha\}) \cup (U - \{x \in Cl_{t+1}^{\succeq} : |D_{R_A}^{+}(x) \cap Cl_t^{\preceq}| \leq k\})$$

$$= (Cl_t^{\preceq} \cup \{x \in Cl_{t+1}^{\succeq} : \frac{|D_{R_A}^{+}(x) \cap Cl_t^{\preceq}|}{|D_{R_A}^{+}(x)|} > 1-\alpha\}) \cup (Cl_t^{\preceq} \cup \{x \in Cl_{t+1}^{\succeq} : |D_{R_A}^{+}(x) \cap Cl_t^{\preceq}| > k\})$$

$$= Cl_t^{\preceq} \cup \{x \in Cl_{t+1}^{\succeq} : \frac{|D_{R_A}^{+}(x) \cap Cl_t^{\preceq}|}{|D_{R_A}^{+}(x)|} > 1-\alpha \vee |D_{R_A}^{+}(x) \cap Cl_t^{\preceq}| > k\}$$

Then the proof process of the above theorem is completed. □

From Definition 5.1 and Theorem 5.1, it is easy to see that $\underline{RA}^{I}_{(\alpha,k)}(Cl_t^{\succeq}) \subseteq Cl_t^{\succeq} \subseteq \overline{RA}^{I}_{(\alpha,k)}(Cl_t^{\succeq})$ and $\underline{RA}^{I}_{(\alpha,k)}(Cl_t^{\preceq}) \subseteq Cl_t^{\preceq} \subseteq \overline{RA}^{I}_{(\alpha,k)}(Cl_t^{\preceq})$.

Proposition 5.1. *Let $(U, AT \cup d, V, f)$ be an ordered information system, $A \subseteq AT$, $t = 1, 2, \cdots, n$, the following properties hold.*

(1) $\underline{RA}^{I}_{(\alpha,k)}(Cl_t^{\succeq}) = \underline{RA}_{\alpha}(Cl_t^{\succeq}) \cap \underline{RA}_{k}(Cl_t^{\succeq})$,
(2) $\overline{RA}^{I}_{(\alpha,k)}(Cl_t^{\succeq}) = \overline{RA}_{\alpha}(Cl_t^{\succeq}) \cup \overline{RA}_{k}(Cl_t^{\succeq})$;
(3) $\underline{RA}^{I}_{(\alpha,k)}(Cl_t^{\preceq}) = \underline{RA}_{\alpha}(Cl_t^{\preceq}) \cap \underline{RA}_{k}(Cl_t^{\preceq})$,
(4) $\overline{RA}^{I}_{(\alpha,k)}(Cl_t^{\preceq}) = \overline{RA}_{\alpha}(Cl_t^{\preceq}) \cup \overline{RA}_{k}(Cl_t^{\preceq})$.

Proof. It can be proved directly from Theorem 5.1. □

All the objects belonging to $Cl_t^{\succeq}$ and $Cl_t^{\preceq}$ with some ambiguity at double-quantitative consistency level $\alpha \in (0, 1]$ and $k \in (0, |U|]$ constitute the boundary regions of $Cl_t^{\succeq}$ and $Cl_t^{\preceq}$.

For $x \in U$, we can define the positive region, negative region and boundary region of $Cl_t^{\succeq}$ as

$$Pos^{I}_{(\alpha,k)}(Cl_t^{\succeq}) = \underline{RA}^{I}_{(\alpha,k)}(Cl_t^{\succeq})$$

$$= \{x \in Cl_t^{\succeq} : \frac{|D_{R_A}^{+}(x) \cap Cl_t^{\succeq}|}{|D_{R_A}^{+}(x)|} \geq \alpha \wedge |D_{R_A}^{+}(x)| - |D_{R_A}^{+}(x) \cap Cl_t^{\succeq}| \leq k\};$$

$$Neg^{I}_{(\alpha,k)}(Cl_t^{\succeq}) = U - \overline{RA}^{I}_{(\alpha,k)}(Cl_t^{\succeq})$$

$$= \{x \in Cl_{t-1}^{\preceq} : \frac{|D_{R_A}^{-}(x) \cap Cl_{t-1}^{\preceq}|}{|D_{R_A}^{-}(x)|} \geq \alpha \wedge |D_{R_A}^{-}(x)| - |D_{R_A}^{-}(x) \cap Cl_{t-1}^{\preceq}| \leq k\};$$

$$Bn^{I}_{(\alpha,k)}(Cl_t^{\succeq}) = \overline{RA}^{I}_{(\alpha,k)}(Cl_t^{\succeq}) - \underline{RA}^{I}_{(\alpha,k)}(Cl_t^{\succeq}).$$

The corresponding three-way decision rules for the upward union $Cl_t^{\succcurlyeq}$ can be obtained:

(P) If $x \in Cl_t^{\succcurlyeq}$ and $\frac{|D_{R_A}^{+}(x) \cap Cl_t^{\succcurlyeq}|}{|D_{R_A}^{+}(x)|} \geq \alpha \wedge |D_{R_A}^{+}(x)| - |D_{R_A}^{+}(x) \cap Cl_t^{\succcurlyeq}| \leq k$, decide $Pos_{(\alpha,k)}^{I}(Cl_t^{\succcurlyeq})$;

(N) If $x \in U - Cl_t^{\succcurlyeq}$ and $\frac{|D_{R_A}^{-}(x) \cap Cl_{t-1}^{\preccurlyeq}|}{|D_{R_A}^{-}(x)|} \geq \alpha \wedge |D_{R_A}^{-}(x)| - |D_{R_A}^{-}(x) \cap Cl_{t-1}^{\preccurlyeq}| \leq k$, decide $Neg_{(\alpha,k)}^{I}(Cl_t^{\succcurlyeq})$;

(B) Otherwise, decide $Bn_{(\alpha,k)}^{I}(Cl_t^{\succcurlyeq})$.

For $x \in U$, the positive region, negative region and boundary region of $Cl_t^{\preccurlyeq}$ are defined as

$$
\begin{aligned}
Pos_{(\alpha,k)}^{I}(Cl_t^{\preccurlyeq}) &= \underline{R_A}_{(\alpha,k)}^{I}(Cl_t^{\preccurlyeq}) \\
&= \{x \in Cl_t^{\preccurlyeq} : \frac{|D_{R_A}^{-}(x) \cap Cl_t^{\preccurlyeq}|}{|D_{R_A}^{-}(x)|} \geq \alpha \wedge |D_{R_A}^{-}(x)| - |D_{R_A}^{-}(x) \cap Cl_t^{\preccurlyeq}| \leq k\}; \\
Neg_{(\alpha,k)}^{I}(Cl_t^{\preccurlyeq}) &= U - \overline{R_A}_{(\alpha,k)}^{I}(Cl_t^{\preccurlyeq}) \\
&= \{x \in Cl_{t+1}^{\succcurlyeq} : \frac{|D_{R_A}^{+}(x) \cap Cl_{t+1}^{\succcurlyeq}|}{|D_{R_A}^{+}(x)|} \geq \alpha \wedge |D_{R_A}^{+}(x)| - |D_{R_A}^{+}(x) \cap Cl_{t+1}^{\succcurlyeq}| \leq k\}; \\
Bn_{(\alpha,k)}^{I}(Cl_t^{\preccurlyeq}) &= \overline{R_A}_{(\alpha,k)}^{I}(Cl_t^{\preccurlyeq}) - \underline{R_A}_{(\alpha,k)}^{I}(Cl_t^{\preccurlyeq}).
\end{aligned}
$$

We can obtain the corresponding three-way decision rules for the downward union $Cl_t^{\preccurlyeq}$:

(P) If $x \in Cl_t^{\preccurlyeq}$ and $\frac{|D_{R_A}^{-}(x) \cap Cl_t^{\preccurlyeq}|}{|D_{R_A}^{-}(x)|} \geq \alpha \wedge |D_{R_A}^{-}(x)| - |D_{R_A}^{-}(x) \cap Cl_t^{\preccurlyeq}| \leq k$, decide $Pos_{(\alpha,k)}^{I}(Cl_t^{\preccurlyeq})$;

(N) If $x \in U - Cl_t^{\preccurlyeq}$ and $\frac{|D_{R_A}^{+}(x) \cap Cl_{t+1}^{\succcurlyeq}|}{|D_{R_A}^{+}(x)|} \geq \alpha \wedge |D_{R_A}^{+}(x)| - |D_{R_A}^{+}(x) \cap Cl_{t+1}^{\succcurlyeq}| \leq k$, decide $Neg_{(\alpha,k)}^{I}(Cl_t^{\preccurlyeq})$;

(B) Otherwise, decide $Bn_{(\alpha,k)}^{I}(Cl_t^{\preccurlyeq})$.

Theorem 5.2. *$\forall t \in T - \{1\}$ and $\forall A \subseteq AT$, $Bn_{(\alpha,k)}^{I}(Cl_t^{\succcurlyeq}) = Bn_{(\alpha,k)}^{I}(Cl_{t-1}^{\preccurlyeq})$.*

Proof. From the definition of the boundary region in DqI-VC-DRSA, we can obtain that

$$
\begin{aligned}
Bn_{(\alpha,k)}^{I}(Cl_t^{\succcurlyeq}) &= \overline{R_A}_{\alpha}(Cl_t^{\succcurlyeq}) - \underline{R_A}_{(\alpha,k)}^{I}(Cl_t^{\succcurlyeq}) \\
&= \overline{R_A}_{(\alpha,k)}^{I}(U - Cl_{t-1}^{\preccurlyeq}) - \underline{R_A}_{(\alpha,k)}^{I}(U - Cl_{t-1}^{\preccurlyeq}) \\
&= U - \underline{R_A}_{(\alpha,k)}^{I}(Cl_{t-1}^{\preccurlyeq}) - (U - \overline{R_A}_{(\alpha,k)}^{I}(Cl_{t-1}^{\preccurlyeq})) \\
&= \overline{R_A}_{(\alpha,k)}^{I}(Cl_{t-1}^{\preccurlyeq}) - \underline{R_A}_{(\alpha,k)}^{I}(Cl_{t-1}^{\preccurlyeq}) \\
&= Bn_{(\alpha,k)}^{I}(Cl_{t-1}^{\preccurlyeq}).
\end{aligned}
$$

Then the proof is completed. □

In DqI-VC-DRSA, the conjunction operator is applied to reflect both the relative quantitative consistency level and absolute quantitative consistency level in the lower approximation. Each element, in the lower approximation, exhibits relative quantification and absolute quantification at the same time with the conjunction operator. However, if the lower approximation contains at least one kind of quantitative consistency level, then the disjunction operator is applied to reflect relative quantitative consistency level or absolute quantitative consistency level. The second kind of Dq-VC-DRSA (denoted as DqII-VC-DRSA) could be studied in the following subsection.

5.2. DqII-VC-DRSA

If the lower approximation contains at least one kind of quantitative consistency level, then the DqII-VC-DRSA in the following Definition 5.2 can be applied.

Definition 5.2. Let $(U, AT \cup d, V, f)$ be an ordered information system, $A \subseteq AT$, $t = 1, 2, \cdots, n$. DqII-VC-DRSA can be defined as follows.

- The second kind of double-quantitative variable consistency dominance-based rough approximations of the upward union $Cl_t^{\succcurlyeq}$ are

$$\begin{aligned}\underline{R_A}^{II}_{(\alpha,k)}(Cl_t^{\succcurlyeq}) =& \{x \in Cl_t^{\succcurlyeq} : \frac{|D^+_{R_A}(x) \cap Cl_t^{\succcurlyeq}|}{|D^+_{R_A}(x)|} \geq \alpha \\ & \vee |D^+_{R_A}(x)| - |D^+_{R_A}(x) \cap Cl_t^{\succcurlyeq}| \leq k\}; \\ \overline{R_A}^{II}_{(\alpha,k)}(Cl_t^{\succcurlyeq}) =& U - \underline{R_A}^{II}_{(\alpha,k)}(U - Cl_t^{\succcurlyeq}).\end{aligned}$$

- The second kind of double-quantitative variable consistency dominance-based rough approximations of the downward union $Cl_t^{\preccurlyeq}$ are

$$\begin{aligned}\underline{R_A}^{II}_{(\alpha,k)}(Cl_t^{\preccurlyeq}) =& \{x \in Cl_t^{\preccurlyeq} : \frac{|D^-_{R_A}(x) \cap Cl_t^{\preccurlyeq}|}{|D^-_{R_A}(x)|} \geq \alpha \\ & \vee |D^-_{R_A}(x)| - |D^-_{R_A}(x) \cap Cl_t^{\preccurlyeq}| \leq k\}; \\ \overline{R_A}^{II}_{(\alpha,k)}(Cl_t^{\preccurlyeq}) =& U - \underline{R_A}^{II}_{(\alpha,k)}(U - Cl_t^{\preccurlyeq}).\end{aligned}$$

Theorem 5.3. *The upper approximations of upward union $Cl_t^{\succcurlyeq}$ and downward union $Cl_t^{\preccurlyeq}$ in DqII-VC-DRSA have the following expressions.*

(1) $\overline{R_A}^{II}_{(\alpha,k)}(Cl_t^{\succcurlyeq}) = Cl_t^{\succcurlyeq} \cup \{x \in Cl_{t-1}^{\preccurlyeq} : \frac{|D^-_{R_A}(x) \cap Cl_t^{\succcurlyeq}|}{|D^-_{R_A}(x)|} > 1 - \alpha \wedge |D^-_{R_A}(x) \cap Cl_t^{\succcurlyeq}| > k\}$;

(2) $\overline{R_A}^{II}_{(\alpha,k)}(Cl_t^{\preccurlyeq}) = Cl_t^{\preccurlyeq} \cup \{x \in Cl_{t+1}^{\succcurlyeq} : \frac{|D^+_{R_A}(x) \cap Cl_t^{\preccurlyeq}|}{|D^+_{R_A}(x)|} > 1 - \alpha \wedge |D^+_{R_A}(x) \cap Cl_t^{\preccurlyeq}| > k\}$.

Proof. (1) From Definition 5.2, the processes about the upper approximation of $Cl_t^{\succcurlyeq}$ in DqII-VC-DRSA are

$$\begin{aligned}\overline{R_A}^{II}_{(\alpha,k)}(Cl_t^{\succcurlyeq}) &= U - \underline{R_A}^{II}_{(\alpha,k)}(U - Cl_t^{\succcurlyeq}) = U - \underline{R_A}^{II}_{(\alpha,k)}(Cl_{t-1}^{\preccurlyeq}) \\ &= U - \{x \in Cl_{t-1}^{\preccurlyeq} : \frac{|D^-_{R_A}(x) \cap Cl_{t-1}^{\preccurlyeq}|}{|D^-_{R_A}(x)|} \geq \alpha \vee |D^-_{R_A}(x)| - |D^-_{R_A}(x) \cap Cl_{t-1}^{\preccurlyeq}| \leq k\} \\ &= U - \{x \in Cl_{t-1}^{\preccurlyeq} : \frac{|D^-_{R_A}(x) \cap Cl_t^{\succcurlyeq}|}{|D^-_{R_A}(x)|} \leq 1 - \alpha \vee |D^-_{R_A}(x) - D^-_{R_A}(x) \cap Cl_{t-1}^{\preccurlyeq}| \leq k\} \\ &= U - \{x \in Cl_{t-1}^{\preccurlyeq} : \frac{|D^-_{R_A}(x) \cap Cl_t^{\succcurlyeq}|}{|D^-_{R_A}(x)|} \leq 1 - \alpha\} \cup \{x \in Cl_{t-1}^{\preccurlyeq} : |D^-_{R_A}(x) \cap Cl_t^{\succcurlyeq}| \leq k\} \\ &= (U - \{x \in Cl_{t-1}^{\preccurlyeq} : \frac{|D^-_{R_A}(x) \cap Cl_t^{\succcurlyeq}|}{|D^-_{R_A}(x)|} \leq 1 - \alpha\}) \cap (U - \{x \in Cl_{t-1}^{\preccurlyeq} : |D^-_{R_A}(x) \cap Cl_t^{\succcurlyeq}| \leq k\}) \\ &= (Cl_t^{\succcurlyeq} \cup \{x \in Cl_{t-1}^{\preccurlyeq} : \frac{|D^-_{R_A}(x) \cap Cl_t^{\succcurlyeq}|}{|D^-_{R_A}(x)|} > 1 - \alpha\}) \cap (Cl_t^{\succcurlyeq} \cup \{x \in Cl_{t-1}^{\preccurlyeq} : |D^-_{R_A}(x) \cap Cl_t^{\succcurlyeq}| > k\}) \\ &= Cl_t^{\succcurlyeq} \cup \{x \in Cl_{t-1}^{\preccurlyeq} : \frac{|D^-_{R_A}(x) \cap Cl_t^{\succcurlyeq}|}{|D^-_{R_A}(x)|} > 1 - \alpha \wedge |D^-_{R_A}(x) \cap Cl_t^{\succcurlyeq}| > k\}\end{aligned}$$

(2) The processes about the upper approximation of $Cl_t^{\preccurlyeq}$ in DqII-VC-DRSA are derived as

$$\begin{aligned}\overline{R_A}^{II}_{(\alpha,k)}(Cl_t^{\preccurlyeq}) &= U - \underline{R_A}^{II}_{(\alpha,k)}(U - Cl_t^{\preccurlyeq}) = U - \underline{R_A}^{II}_{(\alpha,k)}(Cl_{t+1}^{\succcurlyeq}) \\ &= U - \{x \in Cl_{t+1}^{\succcurlyeq} : \frac{|D^+_{R_A}(x) \cap Cl_{t+1}^{\succcurlyeq}|}{|D^+_{R_A}(x)|} \geq \alpha \vee |D^+_{R_A}(x)| - |D^+_{R_A}(x) \cap Cl_{t+1}^{\succcurlyeq}| \leq k\}\end{aligned}$$

$$= U - \{x \in Cl_{t+1}^{\succsim} : \frac{|D_{R_A}^{+}(x) \cap (U - Cl_t^{\precsim})|}{|D_{R_A}^{+}(x)|} \geq \alpha \vee |D_{R_A}^{+}(x) - D_{R_A}^{+}(x) \cap Cl_{t+1}^{\succsim}| \leq k\}$$

$$= U - \{x \in Cl_{t+1}^{\succsim} : \frac{|D_{R_A}^{+}(x) \cap Cl_t^{\precsim}|}{|D_{R_A}^{+}(x)|} \leq 1 - \alpha\} \cup \{x \in Cl_{t+1}^{\succsim} : |D_{R_A}^{+}(x) \cap Cl_t^{\precsim}| \leq k\}$$

$$= (U - \{x \in Cl_{t+1}^{\succsim} : \frac{|D_{R_A}^{+}(x) \cap Cl_t^{\precsim}|}{|D_{R_A}^{+}(x)|} \leq 1 - \alpha\}) \cap (U - \{x \in Cl_{t+1}^{\succsim} : |D_{R_A}^{+}(x) \cap Cl_t^{\precsim}| \leq k\})$$

$$= (Cl_t^{\precsim} \cup \{x \in Cl_{t+1}^{\succsim} : \frac{|D_{R_A}^{+}(x) \cap Cl_t^{\precsim}|}{|D_{R_A}^{+}(x)|} > 1 - \alpha\}) \cap (Cl_t^{\precsim} \cup \{x \in Cl_{t+1}^{\succsim} : |D_{R_A}^{+}(x) \cap Cl_t^{\precsim}| > k\})$$

$$= Cl_t^{\precsim} \cup \{x \in Cl_{t+1}^{\succsim} : \frac{|D_{R_A}^{+}(x) \cap Cl_t^{\precsim}|}{|D_{R_A}^{+}(x)|} > 1 - \alpha \wedge |D_{R_A}^{+}(x) \cap Cl_t^{\precsim}| > k\}$$

Then the proof process of the above theorem is completed. □

From the Definition 5.2 and Theorem 5.3, it is easy to see that $\underline{RA}_{(\alpha,k)}^{II}(Cl_t^{\succsim}) \subseteq Cl_t^{\succsim} \subseteq \overline{RA}_{(\alpha,k)}^{II}(Cl_t^{\succsim})$ and $\underline{RA}_{(\alpha,k)}^{II}(Cl_t^{\precsim}) \subseteq Cl_t^{\precsim} \subseteq \overline{RA}_{(\alpha,k)}^{II}(Cl_t^{\precsim})$.

Proposition 5.2. *Let $(U, AT \cup d, V, f)$ be an ordered information system, $A \subseteq AT$, $t = 1, 2, \cdots, n$, the following properties hold.*

(1) $\underline{RA}_{(\alpha,k)}^{II}(Cl_t^{\succsim}) = \underline{RA}_{\alpha}(Cl_t^{\succsim}) \cup \underline{RA}_{k}(Cl_t^{\succsim})$,
(2) $\overline{RA}_{(\alpha,k)}^{II}(Cl_t^{\succsim}) = \overline{RA}_{\alpha}(Cl_t^{\succsim}) \cap \overline{RA}_{k}(Cl_t^{\succsim})$;
(3) $\underline{RA}_{(\alpha,k)}^{II}(Cl_t^{\precsim}) = \underline{RA}_{\alpha}(Cl_t^{\precsim}) \cup \underline{RA}_{k}(Cl_t^{\precsim})$,
(4) $\overline{RA}_{(\alpha,k)}^{II}(Cl_t^{\precsim}) = \overline{RA}_{\alpha}(Cl_t^{\precsim}) \cap \overline{RA}_{k}(Cl_t^{\precsim})$.

Proof. It can be proved directly from Theorem 5.3. □

All the objects belonging to $Cl_t^{\succsim}$ and $Cl_t^{\precsim}$ with some ambiguity at double-quantitative consistency level $\alpha \in (0, 1]$ and $k \in (0, |U|]$ constitute the boundary regions of $Cl_t^{\succsim}$ and $Cl_t^{\precsim}$.

For $x \in U$, we can define the positive region, negative region and boundary region of $Cl_t^{\succsim}$ as

$$Pos_{(\alpha,k)}^{II}(Cl_t^{\succsim}) = \underline{RA}_{(\alpha,k)}^{II}(Cl_t^{\succsim})$$
$$= \{x \in Cl_t^{\succsim} : \frac{|D_{R_A}^{+}(x) \cap Cl_t^{\succsim}|}{|D_{R_A}^{+}(x)|} \geq \alpha \vee |D_{R_A}^{+}(x)| - |D_{R_A}^{+}(x) \cap Cl_t^{\succsim}| \leq k\};$$
$$Neg_{(\alpha,k)}^{II}(Cl_t^{\succsim}) = U - \overline{RA}_{(\alpha,k)}^{II}(Cl_t^{\succsim})$$
$$= \{x \in Cl_{t-1}^{\precsim} : \frac{|D_{R_A}^{-}(x) \cap Cl_{t-1}^{\precsim}|}{|D_{R_A}^{-}(x)|} \geq \alpha \vee |D_{R_A}^{-}(x)| - |D_{R_A}^{-}(x) \cap Cl_{t-1}^{\precsim}| \leq k\};$$
$$Bn_{(\alpha,k)}^{II}(Cl_t^{\succsim}) = \overline{RA}_{(\alpha,k)}^{II}(Cl_t^{\succsim}) - \underline{RA}_{(\alpha,k)}^{I}(Cl_t^{\succsim}).$$

The corresponding three-way decision rules for the upward union $Cl_t^{\succsim}$ can be obtained:

(P) If $x \in Cl_t^{\succsim}$ and $\frac{|D_{R_A}^{+}(x) \cap Cl_t^{\succsim}|}{|D_{R_A}^{+}(x)|} \geq \alpha \vee |D_{R_A}^{+}(x)| - |D_{R_A}^{+}(x) \cap Cl_t^{\succsim}| \leq k$, decide $Pos_{(\alpha,k)}^{II}(Cl_t^{\succsim})$;
(N) If $x \in U - Cl_t^{\succsim}$ and $\frac{|D_{R_A}^{-}(x) \cap Cl_{t-1}^{\precsim}|}{|D_{R_A}^{-}(x)|} \geq \alpha \vee |D_{R_A}^{-}(x)| - |D_{R_A}^{-}(x) \cap Cl_{t-1}^{\precsim}| \leq k$, decide $Neg_{(\alpha,k)}^{II}(Cl_t^{\succsim})$;
(B) Otherwise, decide $Bn_{(\alpha,k)}^{II}(Cl_t^{\succsim})$.

Table 5.1
An ordered information system.

U	a_1	a_2	d
x_1	3	3	3
x_2	2	2	3
x_3	1	3	3
x_4	3	1	2
x_5	4	4	1

For $x \in U$, the positive region, negative region and boundary region of $Cl_t^{\preccurlyeq}$ are defined as

$$\begin{aligned} Pos^{II}_{(\alpha,k)}(Cl_t^{\preccurlyeq}) &= \underline{R_A}^{II}_{(\alpha,k)}(Cl_t^{\preccurlyeq}) \\ &= \{x \in Cl_t^{\preccurlyeq} : \frac{|D^-_{R_A}(x) \cap Cl_t^{\preccurlyeq}|}{|D^-_{R_A}(x)|} \geq \alpha \vee |D^-_{R_A}(x)| - |D^-_{R_A}(x) \cap Cl_t^{\preccurlyeq}| \leq k\}; \\ Neg^{II}_{(\alpha,k)}(Cl_t^{\preccurlyeq}) &= U - \overline{R_A}^{II}_{(\alpha,k)}(Cl_t^{\preccurlyeq}) \\ &= \{x \in Cl_{t+1}^{\succcurlyeq} : \frac{|D^+_{R_A}(x) \cap Cl_{t+1}^{\succcurlyeq}|}{|D^+_{R_A}(x)|} \geq \alpha \vee |D^+_{R_A}(x)| - |D^+_{R_A}(x) \cap Cl_{t+1}^{\succcurlyeq}| \leq k\}; \\ Bn^{II}_{(\alpha,k)}(Cl_t^{\preccurlyeq}) &= \overline{R_A}^{II}_{(\alpha,k)}(Cl_t^{\preccurlyeq}) - \underline{R_A}^{II}_{(\alpha,k)}(Cl_t^{\preccurlyeq}). \end{aligned}$$

We can obtain the corresponding three-way decision rules for the downward union $Cl_t^{\preccurlyeq}$:

(P) If $x \in Cl_t^{\preccurlyeq}$ and $\frac{|D^-_{R_A}(x) \cap Cl_t^{\preccurlyeq}|}{|D^-_{R_A}(x)|} \geq \alpha \vee |D^-_{R_A}(x)| - |D^-_{R_A}(x) \cap Cl_t^{\preccurlyeq}| \leq k$, decide $Pos^{II}_{(\alpha,k)}(Cl_t^{\preccurlyeq})$;

(N) If $x \in U - Cl_t^{\preccurlyeq}$ and $\frac{|D^+_{R_A}(x) \cap Cl_{t+1}^{\succcurlyeq}|}{|D^+_{R_A}(x)|} \geq \alpha \vee |D^+_{R_A}(x)| - |D^+_{R_A}(x) \cap Cl_{t+1}^{\succcurlyeq}| \leq k$, decide $Neg^{II}_{(\alpha,k)}(Cl_t^{\preccurlyeq})$;

(B) Otherwise, decide $Bn^{II}_{(\alpha,k)}(Cl_t^{\preccurlyeq})$.

Theorem 5.4. *$\forall t \in T - \{1\}$ and $\forall A \subseteq AT$, $Bn^{II}_{(\alpha,k)}(Cl_t^{\succcurlyeq}) = Bn^{II}_{(\alpha,k)}(Cl_{t-1}^{\preccurlyeq})$.*

Proof. From the definition of the boundary region in DqII-VC-DRSA, we can obtain that

$$\begin{aligned} Bn^{II}_{(\alpha,k)}(Cl_t^{\succcurlyeq}) &= \overline{R_A}^{II}_{(\alpha,k)}(Cl_t^{\succcurlyeq}) - \underline{R_A}^{II}_{(\alpha,k)}(Cl_t^{\succcurlyeq}) \\ &= \overline{R_A}^{II}_{(\alpha,k)}(U - Cl_{t-1}^{\preccurlyeq}) - \underline{R_A}^{II}_{(\alpha,k)}(U - Cl_{t-1}^{\preccurlyeq}) \\ &= U - \underline{R_A}^{II}_{(\alpha,k)}(Cl_{t-1}^{\preccurlyeq}) - (U - \overline{R_A}^{II}_{(\alpha,k)}(Cl_{t-1}^{\preccurlyeq})) \\ &= \overline{R_A}^{II}_{(\alpha,k)}(Cl_{t-1}^{\preccurlyeq}) - \underline{R_A}^{II}_{(\alpha,k)}(Cl_{t-1}^{\preccurlyeq}) \\ &= Bn^{II}_{(\alpha,k)}(Cl_{t-1}^{\preccurlyeq}). \end{aligned}$$

Then the proof is completed. □

In the reference [1], authors addressed basic significant monotonicity properties related to DRSA, which are (m1) monotonicity with respect to the set of attributes; (m2) monotonicity with respect to the set of objects; (m3) monotonicity with respect to the union of ordered classes; (m4) monotonicity with respect to the dominance classes.

It has been shown that these properties exhibit important influence on the rule induction for variable consistency rough set approaches [2]. For this reason, it is necessary to verify whether the monotonicity properties mentioned are applicable to single-quantitative and double-quantitative variable consistency dominance-based rough approximations in this paper.

In the Sq-VC-DRSA models, both relative and absolute quantitative consistency levels of $Cl_t^{\succcurlyeq}$ (or $Cl_t^{\preccurlyeq}$) have properties (m2) and (m3), which can be proved from the item (4) of Theorem 4.4 and Theorem 4.8, respectively. Unfortunately, they have neither property (m1) or (m4). We explain it by the ordered information system (see Table 5.1) considered in the literature [1]. For the sake of simple expression, we use $r^{A_j}_{Cl_t^{\succcurlyeq}}(x_i)$ to denote the relative quantitative consistency measure $|D^+_{R_{A_j}}(x_i) \cap Cl_t^{\succcurlyeq}|/|D^+_{R_{A_j}}(x_i)|$, and use $a^{A_j}_{Cl_t^{\succcurlyeq}}(x_i)$ to denote the absolute quantitative consistency measure $|D^+_{R_{A_j}}(x_i)| - |D^+_{R_{A_j}}(x_i) \cap Cl_t^{\succcurlyeq}|$. Let us consider $Cl_3^{\succcurlyeq} = \{x_1, x_2, x_3\}$, the attribute sets $A_1 = \{a_1\}$, $A_2 = \{a_2\}$ $A_3 = \{a_1, a_2\}$, it is easy to see that $A_1, A_2 \subseteq$

A_3. For the Rq-VC-DRSA, we calculate that $r^{A_2}_{Cl_3^{\succeq}}(x_2) = 3/4$, while $r^{A_3}_{Cl_3^{\succeq}}(x_2) = 2/3$. Since $r^{A_2}_{Cl_3^{\succeq}}(x_2) > r^{A_3}_{Cl_3^{\succeq}}(x_2)$, the relative quantitative consistency measure does not satisfy the property (m1). In addition, from $x_1 \in D^+_{R_{A_3}}(x_2)$ and $r^{A_3}_{Cl_3^{\succeq}}(x_1) = 1/2 < r^{A_3}_{Cl_3^{\succeq}}(x_2) = 2/3$, we obtain that the relative quantitative consistency measure does not satisfy the property (m4). For the Aq-VC-DRSA, $a^{A_1}_{Cl_3^{\succeq}}(x_1) = 2$, while $a^{A_3}_{Cl_3^{\succeq}}(x_1) = 1$, the absolute quantitative consistency measure does not satisfy the property (m1). In addition, $x_5 \in D^+_{R_{A_3}}(x_1)$ and $a^{A_3}_{Cl_3^{\succeq}}(x_5) = 0 < a^{A_3}_{Cl_3^{\succeq}}(x_1) = 2$, so the absolute quantitative consistency measure does not satisfy the property (m4).

As to the Dq-VC-DRSA models, according to Proposition 5.1 and Proposition 5.2 about the relationship between double-quantitative and single-quantitative consistency rough approximations, it can be verified that the double-quantitative consistency measures used in the models have the properties (m2) and (m3), but do not hold the properties (m1) and (m4).

6. Comparison and analysis

There are many partial ordered decision-making problems about evaluation in real life. In the reference [13], authors used an example about customer satisfaction with airline services to explain the VC-DRSA. Inspired by reference [13], here we assume that an airline has also conducted a questionnaire on the service quality for its customers in order to evaluate the company's services and then update the equipment. According to Dq-VC-DRSA models, we can make more comprehensive evaluations in the actual decision-making processes.

Table 6.1 is an ordered information system about the questionnaire diffused by the assumed airline. The table contains 20 objects (customers' feedbacks) described by $U = \{x_1, x_2, \cdots, x_{20}\}$ of criteria corresponding to the considered three aspects of the aircraft comfort: space for seat width (SW), hand luggage (HL), and leg room (LR). There is also an item about an overall evaluation d. One uses 1, 2, 3 to denote the customer's satisfaction values of the three aspects SW, HL, LR and the overall evaluation d, where the numbers 1, 2, 3 are respectively for Bad, Medium, Good.

The overall evaluation d creates three decision classes, which are preference ordered according to increasing class number, i.e. $Cl_1 = $ Bad, $Cl_2 = $ Medium, $Cl_3 = $ Good. The decision classes are shown as follows.

$$Cl_1 = \{x_2, x_4, x_6, x_{10}, x_{11}, x_{17}, x_{18}, x_{19}\};$$

$$Cl_2 = \{x_1, x_3, x_5, x_7, x_{12}, x_{13}, x_{14}, x_{15}\};$$

$$Cl_3 = \{x_8, x_9, x_{16}, x_{20}\}.$$

As the above decision classes are ordered, the following downward and upward unions of decision classes are to be considered.

(1) *At most bad*: $Cl_1^{\preceq} = \{x_2, x_4, x_6, x_{10}, x_{11}, x_{17}, x_{18}, x_{19}\}$.

(2) *At most medium*: $Cl_2^{\preceq} = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{17}, x_{18}, x_{19}\}$.

(3) *At least medium*: $Cl_2^{\succeq} = \{x_1, x_3, x_5, x_7, x_8, x_9, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}, x_{20}\}$.

(4) *At least good*: $Cl_3^{\succeq} = \{x_8, x_9, x_{16}, x_{20}\}$.

From the data described in Table 6.1, we obtain all inconsistencies in the downward union and upward union of decision classes. The inconsistency situation for the data described in Table 6.1 can be shown in Table 6.2. There are many possible reasons for customers about these inconsistencies in the evaluation of airline comfort, such as

(1) Different customers have different heights, weights and other physical indicators, which leads to different feelings about the space on the aircraft;
(2) Different customers have different boundaries between good and bad;

Table 6.1
Statistical results.

U	SW	HL	LR	d	U	SW	HL	LR	d
x_1	1	3	2	2	x_{11}	3	2	2	1
x_2	3	3	1	1	x_{12}	1	2	3	2
x_3	1	3	2	2	x_{13}	3	2	2	2
x_4	3	1	3	1	x_{14}	2	3	1	2
x_5	2	3	1	2	x_{15}	3	2	1	2
x_6	3	3	1	1	x_{16}	1	2	2	3
x_7	1	3	3	2	x_{17}	3	3	1	1
x_8	1	3	3	3	x_{18}	3	1	2	1
x_9	3	2	1	3	x_{19}	2	2	3	1
x_{10}	2	2	2	1	x_{20}	3	2	1	3

Table 6.2
Inconsistency description.

	$D^+_{R_A}(x)$	Inconsistency situation	$D^-_{R_A}(x)$	Inconsistency situation
$x_1 \in Cl_2$	$x_{1,3,7,8}$		$x_{1,3,16}$	$x_{16} \in Cl_3^{\succcurlyeq}$
$x_2 \in Cl_1$	$x_{2,6,17}$		$x_{2,5,6,9,14,15,17,20}$	$x_{5,9,14,15,20} \in Cl_2^{\succcurlyeq}$
$x_3 \in Cl_2$	$x_{1,3,7,8}$		$x_{1,3,16}$	$x_{16} \in Cl_3^{\succcurlyeq}$
$x_4 \in Cl_1$	x_4		$x_{4,18}$	
$x_5 \in Cl_2$	$x_{2,5,6,14,17}$	$x_{2,6,17} \in Cl_1^{\preccurlyeq}$	$x_{5,14}$	
$x_6 \in Cl_1$	$x_{2,6,17}$		$x_{2,5,6,9,14,15,17,20}$	$x_{5,9,14,15,20} \in Cl_2^{\succcurlyeq}$
$x_7 \in Cl_2$	$x_{7,8}$		$x_{1,3,7,8,12,16}$	$x_{8,16} \in Cl_3^{\succcurlyeq}$
$x_8 \in Cl_3$	$x_{7,8}$		$x_{1,3,7,8,12,16}$	
$x_9 \in Cl_3$	$x_{2,6,9,11,13,15,17,20}$	$x_{2,6,11,15,17} \in Cl_2^{\preccurlyeq}$	$x_{9,15,20}$	
$x_{10} \in Cl_1$	$x_{10,11,13,19}$		$x_{10,16}$	$x_{16} \in Cl_3^{\succcurlyeq}$
$x_{11} \in Cl_1$	$x_{11,13}$		$x_{9,10,11,13,15,16,18,20}$	$x_{9,13,15,16,20} \in Cl_2^{\succcurlyeq}$
$x_{12} \in Cl_2$	$x_{7,8,12,19}$	$x_{19} \in Cl_1^{\preccurlyeq}$	$x_{12,16}$	$x_{16} \in Cl_3^{\succcurlyeq}$
$x_{13} \in Cl_2$	$x_{11,13}$		$x_{9,10,11,13,15,16,18,20}$	$x_{9,16,20} \in Cl_3^{\succcurlyeq}$
$x_{14} \in Cl_2$	$x_{2,5,6,14,17}$		$x_{5,14}$	
$x_{15} \in Cl_2$	$x_{2,6,9,11,13,15,17,20}$		$x_{9,15,20}$	
$x_{16} \in Cl_3$	$x_{1,3,7,8,10,11,12,13,16,19}$	$x_{1,3,7,10,11,12,13,19} \in Cl_2^{\preccurlyeq}$	x_{16}	
$x_{17} \in Cl_1$	$x_{2,6,17}$		$x_{2,5,6,9,14,15,17,20}$	$x_{5,9,14,15,20} \in Cl_2^{\succcurlyeq}$
$x_{18} \in Cl_1$	$x_{4,11,13,18}$		x_{18}	
$x_{19} \in Cl_1$	x_{19}		$x_{10,12,16,19}$	$x_{12,16} \in Cl_2^{\succcurlyeq}$
$x_{20} \in Cl_3$	$x_{2,6,9,11,13,15,17,20}$		$x_{9,15,20}$	

(3) Different customers have different expectations for the airline;
(4) Errors in the recording of data, and others.

Let us take the object $x_{12} \in Cl_2$ for example: (1) $D^+_{R_A}(x_{12}) = \{x_7, x_8, x_{12}, x_{19}\}$, which means x_{12} is dominated by x_7, x_8, x_{12} and x_{19}. It is easy to see that $x_{19} \in Cl_1^{\preccurlyeq}$, while x_{12} is assigned to the decision class Cl_2 better than $Cl_1^{\preccurlyeq}$. (2) $D^-_{R_A}(x_{12}) = \{x_{12}, x_{16}\}$, which means x_{12} dominates x_{12} and x_{16}. It is easy to see that $x_{16} \in Cl_3^{\succcurlyeq}$, while x_{12} is assigned to the decision class Cl_2 worse than $Cl_3^{\succcurlyeq}$. This means that x_{12} gave an evaluation for all the considered aspects not worse than the evaluation given by x_{16}, while x_{12} gave an overall evaluation of the aircraft comfort worse than the overall evaluation of x_{16}; x_{12} gave an evaluation for all the considered aspects worse than the evaluation given by x_{19}, while gave an overall evaluation of the aircraft comfort not worse than the overall evaluation of x_{19}.

Next we will analyze which customers' overall evaluation is consistent with their scores on the three considered indicators, which customers' overall evaluation is totally inconsistent with their scores of the three indicators, and which customers' overall evaluation cannot be determined whether they are consistent with their scores of three indicators based on the scores they provided. Then computing the proposed Sq-VC-DRSA and Dq-VC-DRSA is natural. Suppose the relative quantitative consistency level $\alpha = 0.7$ and the absolute quantitative consistency level $k = 2$.

In order to facilitate the comparisons, we first calculate the lower and upper approximations of $Cl_1^{\preccurlyeq}$ in DRSA as

$$\begin{aligned}\underline{R_A}(Cl_1^{\preccurlyeq}) =& \{x_4, x_{18}\};\\ \overline{R_A}(Cl_1^{\preccurlyeq}) =& \{x_2, x_4, x_5, x_6, x_9, x_{10}, x_{11}, x_{12},\\ & x_{13}, x_{14}, x_{15}, x_{16}, x_{17}, x_{18}, x_{19}, x_{20}\}.\end{aligned}$$

The lower and upper approximations of $Cl_2^{\preccurlyeq}$ in DRSA are

$$\begin{aligned}\underline{R_A}(Cl_2^{\preccurlyeq}) &= \{x_4, x_5, x_{14}, x_{18}\};\\ \overline{R_A}(Cl_2^{\preccurlyeq}) &= U.\end{aligned}$$

The lower and upper approximations of $Cl_2^{\succcurlyeq}$ in DRSA are

$$\begin{aligned}\underline{R_A}(Cl_2^{\succcurlyeq}) =& \{x_1, x_3, x_7, x_8\};\\ \overline{R_A}(Cl_2^{\succcurlyeq}) =& \{x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_9, x_{10},\\ & x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}, x_{17}, x_{19}, x_{20}\}.\end{aligned}$$

The lower and upper approximations of $Cl_3^{\succcurlyeq}$ in DRSA are

$$\underline{R_A}(Cl_3^{\succcurlyeq}) = \emptyset;$$
$$\overline{R_A}(Cl_3^{\succcurlyeq}) = \{x_1, x_2, x_3, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{15}, x_{16}, x_{17}, x_{19}, x_{20}\}.$$

With the DRSA, elements belonging to the positive region mean that the dominating set (dominated set) of these objects is consistent to the upward union (or downward union); elements belonging to the negative region mean that the dominating set (dominated set) of these objects is totally inconsistent with the upward union (or downward union); elements belonging to the boundary region mean that the dominating set (dominated set) of these objects cannot be judged whether the dominating set (dominated set) is consistent with the upward union (or downward union). However, for the DRSA, the conditions imposed on the relationship between dominating set (dominated set) and upward union (downward union) are so strict that there are no fault tolerance mechanisms. Quantitative information about the degree of overlap of the dominating set (dominated set) and upward union (downward union) is not taken into consideration. In fact, we could allow a certain degree of inconsistencies to exist in real-life applications. Just as presented in Section 3, two kinds of quantitative consistency levels could be obtained from the two kinds of quantitative information, namely relative quantitative level and absolute quantitative level. In the following, we will discuss the three-way decision regions in Sq-VC-DRSA models and Dq-VC-DRSA models. Table 6.3 and Table 6.4 are about the statistical results of upward union $Cl_t^{\succcurlyeq}$ and downward union $Cl_t^{\preccurlyeq}$ $(t = 1, 2, 3)$, respectively.

Based on the data stated in Table 6.3 and Table 6.4, we calculate the lower and upper approximations in Rq-VC-DRSA as follows.

The lower and upper approximations of downward union $Cl_1^{\preccurlyeq}$ in Rq-VC-DRSA are

$$\underline{R_{A_\alpha}}(Cl_1^{\preccurlyeq}) = \{x_4, x_{18}\};$$
$$\overline{R_{A\alpha}}(Cl_1^{\preccurlyeq}) = \{x_2, x_4, x_5, x_6, x_9, x_{10}, x_{11}, x_{13}, x_{14}, x_{15}, x_{17}, x_{18}, x_{19}, x_{20}\}.$$

The lower and upper approximations of downward union $Cl_2^{\preccurlyeq}$ in Rq-VC-DRSA are

$$\underline{R_{A_\alpha}}(Cl_2^{\preccurlyeq}) = \{x_2, x_4, x_5, x_6, x_{14}, x_{17}, x_{18}, x_{19}\};$$
$$\overline{R_{A\alpha}}(Cl_2^{\preccurlyeq}) = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}, x_{17}, x_{18}, x_{19}, x_{20}\}.$$

The lower and upper approximations of upward union $Cl_2^{\succcurlyeq}$ in Rq-VC-DRSA are

$$\underline{R_{A_\alpha}}(Cl_2^{\succcurlyeq}) = \{x_1, x_3, x_7, x_8, x_{12}, x_{16}\};$$
$$\overline{R_{A\alpha}}(Cl_2^{\succcurlyeq}) = \{x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}, x_{17}, x_{19}, x_{20}\}.$$

The lower and upper approximations of upward union $Cl_3^{\succcurlyeq}$ in Rq-VC-DRSA are

$$\underline{R_{A_\alpha}}(Cl_3^{\succcurlyeq}) = \emptyset;$$
$$\overline{R_{A\alpha}}(Cl_3^{\succcurlyeq}) = \{x_1, x_3, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{15}, x_{16}, x_{20}\}.$$

And the lower and upper approximations in Aq-VC-DRSA are calculated as follows.

The lower and upper approximations of downward union $Cl_1^{\preccurlyeq}$ in Aq-VC-DRSA are

$$\underline{R_{A_k}}(Cl_1^{\preccurlyeq}) = \{x_4, x_{10}, x_{18}, x_{19}\};$$
$$\overline{R_{Ak}}(Cl_1^{\preccurlyeq}) = \{x_2, x_4, x_5, x_6, x_9, x_{10}, x_{11}, x_{14}, x_{15}, x_{16}, x_{17}, x_{18}, x_{19}, x_{20}\}.$$

Table 6.3
Statistical results of upward union $Cl_t^{\succcurlyeq}$.

U	$\lvert D^+_{R_A}(x)\rvert$	$\lvert D^+_{R_A}(x)\cap Cl_t^{\succcurlyeq}\rvert$		$\frac{\lvert D^+_{R_A}(x)\cap Cl_t^{\succcurlyeq}\rvert}{\lvert D^+_{R_A}(x)\rvert}$		$\lvert D^+_{R_A}(x)\rvert-\lvert D^+_{R_A}(x)\cap Cl_t^{\succcurlyeq}\rvert$		$\lvert D^-_{R_A}(x)\rvert$	$\lvert D^-_{R_A}(x)\cap Cl_t^{\succcurlyeq}\rvert$		$\frac{\lvert D^-_{R_A}(x)\cap Cl_t^{\succcurlyeq}\rvert}{\lvert D^-_{R_A}(x)\rvert}$		$\lvert D^-_{R_A}(x)\rvert-\lvert D^-_{R_A}(x)\cap Cl_t^{\succcurlyeq}\rvert$	
		$t=2$	$t=3$	$t=2$	$t=3$	$t=2$	$t=3$		$t=2$	$t=3$	$t=2$	$t=3$	$t=2$	$t=3$
x_1	4	4	1	1	1/4	0	3	3	3	1	1	1/3	0	2
x_2	3	0	0	0	0	3	3	8	5	2	5/8	2/8	3	6
x_3	4	4	1	1	1/4	0	3	3	3	1	1	1/3	0	2
x_4	1	0	0	0	0	1	1	2	0	0	0	0	2	2
x_5	5	2	0	2/5	0	3	5	2	2	0	1	0	0	2
x_6	3	0	0	0	0	3	3	8	5	2	5/8	2/8	3	6
x_7	2	2	1	1	1/2	0	1	6	6	2	1	2/6	0	4
x_8	2	2	1	1	1/2	0	1	6	6	2	1	2/6	0	4
x_9	8	4	2	4/8	2/8	4	6	3	3	2	1	2/3	0	1
x_{10}	4	1	0	1/4	0	3	4	2	1	1	1/2	1/2	1	1
x_{11}	2	1	0	1/2	0	1	2	8	5	3	5/8	3/8	3	5
x_{12}	4	3	1	3/4	1/4	1	3	2	2	1	1	1/2	0	1
x_{13}	2	1	0	1/2	0	1	2	8	5	3	5/8	3/8	3	5
x_{14}	5	2	0	2/5	0	3	5	2	2	0	1	0	0	2
x_{15}	8	4	2	4/8	2/8	4	6	3	3	2	1	2/3	0	1
x_{16}	10	7	2	7/10	2/10	3	8	1	1	1	1	1	0	0
x_{17}	3	0	0	0	0	3	3	8	5	2	5/8	2/8	3	6
x_{18}	4	1	0	1/4	0	3	4	1	0	0	0	0	1	1
x_{19}	1	0	0	0	0	1	1	4	2	1	1/2	1/4	2	3
x_{20}	8	4	2	4/8	2/8	4	6	3	3	2	1	2/3	0	1

Table 6.4
Statistical results of downward union $Cl_t^{\preccurlyeq}$.

U	$\|D^+_{R_A}(x)\|$	$\|D^+_{R_A}(x) \cap Cl_t^{\preccurlyeq}\|$		$\frac{\|D^+_{R_A}(x) \cap Cl_t^{\preccurlyeq}\|}{\|D^+_{R_A}(x)\|}$		$\|D^+_{R_A}(x)\| - \|D^+_{R_A}(x) \cap Cl_t^{\preccurlyeq}\|$		$\|D^-_{R_A}(x)\|$	$\|D^-_{R_A}(x) \cap Cl_t^{\preccurlyeq}\|$		$\frac{\|D^-_{R_A}(x) \cap Cl_t^{\preccurlyeq}\|}{\|D^-_{R_A}(x)\|}$		$\|D^-_{R_A}(x)\| - \|D^-_{R_A}(x) \cap Cl_t^{\preccurlyeq}\|$	
		$t=1$	$t=2$	$t=1$	$t=2$	$t=1$	$t=2$		$t=1$	$t=2$	$t=1$	$t=2$	$t=1$	$t=2$
x_1	4	0	3	0	3/4	4	1	3	0	2	0	2/3	3	1
x_2	3	3	3	1	1	0	0	8	3	6	3/8	6/8	5	2
x_3	4	0	3	0	3/4	4	1	3	0	2	0	2/3	3	1
x_4	1	1	1	1	1	0	0	2	2	2	1	1	0	0
x_5	5	3	5	3/5	1	2	0	2	0	2	0	1	2	0
x_6	3	3	3	1	1	0	0	8	3	6	3/8	6/8	5	2
x_7	2	0	1	0	1/2	2	1	6	0	4	0	4/6	6	2
x_8	2	0	1	0	1/2	2	1	6	0	4	0	4/6	6	2
x_9	8	4	6	4/8	6/8	4	2	3	0	1	0	1/3	3	2
x_{10}	4	3	4	3/4	1	1	0	2	1	1	1/2	1/2	1	1
x_{11}	2	1	2	1/2	1	1	0	8	3	5	3/8	5/8	5	3
x_{12}	4	1	3	1/4	3/4	3	1	2	0	1	0	1/2	2	1
x_{13}	2	1	2	1/2	1	1	0	8	3	5	3/8	5/8	5	3
x_{14}	5	3	5	3/5	1	2	0	2	0	2	0	1	2	0
x_{15}	8	4	6	4/8	6/8	4	2	3	0	1	0	1/3	3	2
x_{16}	10	3	8	3/10	8/10	7	2	1	0	0	0	0	1	1
x_{17}	3	3	3	1	1	0	0	8	3	6	3/8	6/8	5	2
x_{18}	4	3	4	3/4	1	1	0	1	1	1	1	1	0	0
x_{19}	1	1	1	1	1	0	0	4	2	3	2/4	3/4	2	1
x_{20}	8	4	6	4/8	6/8	4	2	3	0	1	0	1/3	3	2

The lower and upper approximations of downward union $Cl_2^{\preccurlyeq}$ in Aq-VC-DRSA are

$$\begin{aligned}\underline{R_{A_k}}(Cl_2^{\preccurlyeq}) =&\{x_1, x_2, x_3, x_4, x_5, x_6, x_7,\\ &x_{10}, x_{12}, x_{14}, x_{15}, x_{17}, x_{18}, x_{19}\};\\ \overline{R_{A_k}}(Cl_2^{\preccurlyeq}) =&\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_9, x_{10}, x_{11},\\ &x_{12}, x_{13}, x_{14}, x_{15}, x_{16}, x_{17}, x_{18}, x_{19}, x_{20}\}.\end{aligned}$$

The lower and upper approximations of upward union $Cl_2^{\succcurlyeq}$ in Aq-VC-DRSA are

$$\begin{aligned}\underline{R_{A_k}}(Cl_2^{\succcurlyeq}) =&\{x_1, x_3, x_7, x_8, x_{12}, x_{13}\};\\ \overline{R_{A_k}}(Cl_2^{\succcurlyeq}) =&\{x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_9,\\ &x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}, x_{17}, x_{20}\}.\end{aligned}$$

The lower and upper approximations of upward union $Cl_3^{\succcurlyeq}$ in Aq-VC-DRSA are

$$\begin{aligned}\underline{R_{A_k}}(Cl_3^{\succcurlyeq}) &= \{x_8\};\\ \overline{R_{A_k}}(Cl_3^{\succcurlyeq}) &= \{x_8, x_9, x_{11}, x_{13}, x_{16}, x_{20}\}.\end{aligned}$$

The lower and upper approximations in DqI-VC-DRSA are calculated as follows.
The lower and upper approximations of downward union $Cl_1^{\preccurlyeq}$ in DqI-VC-DRSA are

$$\begin{aligned}\underline{R_A}^{I}_{(\alpha,k)}(Cl_1^{\preccurlyeq}) =&\{x_4, x_{18}\};\\ \overline{R_A}^{I}_{(\alpha,k)}(Cl_1^{\preccurlyeq}) =&\{x_2, x_4, x_5, x_6, x_9, x_{10}, x_{11},\\ &x_{13}, x_{14}, x_{15}, x_{16}, x_{17}, x_{18}, x_{19}, x_{20}\}.\end{aligned}$$

The lower and upper approximations of downward union $Cl_2^{\preccurlyeq}$ in DqI-VC-DRSA are

$$\begin{aligned}\underline{R_A}^{I}_{(\alpha,k)}(Cl_2^{\preccurlyeq}) =&\{x_2, x_4, x_5, x_6, x_{14}, x_{17}, x_{18}, x_{19}\};\\ \overline{R_A}^{I}_{(\alpha,k)}(Cl_2^{\preccurlyeq}) =&\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10},\\ &x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}, x_{17}, x_{18}, x_{19}, x_{20}\}.\end{aligned}$$

The lower and upper approximations of upward union $Cl_2^{\succcurlyeq}$ in DqI-VC-DRSA are

$$\begin{aligned}\underline{R_A}^{I}_{(\alpha,k)}(Cl_2^{\succcurlyeq}) =&\{x_1, x_3, x_7, x_8, x_{12}\};\\ \overline{R_A}^{I}_{(\alpha,k)}(Cl_2^{\succcurlyeq}) =&\{x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_9, x_{10},\\ &x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}, x_{17}, x_{19}, x_{20}\}.\end{aligned}$$

The lower and upper approximations of upward union $Cl_3^{\succcurlyeq}$ in DqI-VC-DRSA are

$$\begin{aligned}\underline{R_A}^{I}_{(\alpha,k)}(Cl_3^{\succcurlyeq}) =&\emptyset;\\ \overline{R_A}^{I}_{(\alpha,k)}(Cl_3^{\succcurlyeq}) =&\{x_1, x_3, x_7, x_8, x_9, x_{10},\\ &x_{11}, x_{12}, x_{13}, x_{15}, x_{16}, x_{20}\}.\end{aligned}$$

The lower and upper approximations in DqII-VC-DRSA are calculated as follows.
The lower and upper approximations of downward union $Cl_1^{\preccurlyeq}$ in DqII-VC-DRSA are

$$\begin{aligned}\underline{R_A}^{II}_{(\alpha,k)}(Cl_1^{\preccurlyeq}) =&\{x_4, x_{10}, x_{18}, x_{19}\};\\ \overline{R_A}^{II}_{(\alpha,k)}(Cl_1^{\preccurlyeq}) =&\{x_2, x_4, x_5, x_6, x_9, x_{10},\\ &x_{11}, x_{14}, x_{15}, x_{17}, x_{18}, x_{19}, x_{20}\}.\end{aligned}$$

Table 6.5
Decision regions of downward union $Cl_1^{\preccurlyeq}$ in different models.

Model		Positive region	Negative region	Boundary region
DRSA		$x_{4,18}$	$x_{1,3,7,8}$	$x_{2,5,6,9,10,11,12,13,14,15,16,17,19,20}$
Rq-VC-DRSA		$x_{4,18}$	$x_{1,3,7,8,12,16}$	$x_{2,5,6,9,10,11,13,14,15,17,19,20}$
Aq-VC-DRSA		$x_{4,10,18,19}$	$x_{1,3,7,8,12,13}$	$x_{2,5,6,9,11,14,15,16,17,20}$
Dq-VC-DRSA	I	$x_{4,18}$	$x_{1,3,7,8,12}$	$x_{2,5,6,9,10,11,13,14,15,16,17,19,20}$
	II	$x_{4,10,18,19}$	$x_{1,3,7,8,12,13,16}$	$x_{2,5,6,9,11,14,15,17,20}$

Table 6.6
Decision regions of downward union $Cl_2^{\preccurlyeq}$ in different models.

Model		Positive region	Negative region	Boundary region
DRSA		$x_{4,5,14,18}$	$\emptyset$	$x_{1,2,3,6,7,8,9,10,11,12,13,15,16,17,19,20}$
Rq-VC-DRSA		$x_{2,4,5,6,14,17,18,19}$	$\emptyset$	$x_{1,3,7,8,9,10,11,12,13,15,16,20}$
Aq-VC-DRSA		$x_{1,2,3,4,5,6,7,10,12,14,15,17,18,19}$	x_8	$x_{9,11,13,16,20}$
Dq-VC-DRSA	I	$x_{2,4,5,6,14,17,18,19}$	$\emptyset$	$x_{1,3,7,8,9,10,11,12,13,15,16,20}$
	II	$x_{1,2,3,4,5,6,7,10,12,14,15,17,18,19}$	x_8	$x_{9,11,13,16,20}$

Table 6.7
Decision regions of upward union $Cl_2^{\succcurlyeq}$ in different model.

Model		Positive region	Negative region	Boundary region
DRSA		$x_{1,3,7,8}$	$x_{4,18}$	$x_{2,5,6,9,10,11,12,13,14,15,16,17,19,20}$
Rq-VC-DRSA		$x_{1,3,7,8,12,16}$	$x_{4,18}$	$x_{2,5,6,9,10,11,13,14,15,17,19,20}$
Aq-VC-DRSA		$x_{1,3,7,8,12,13}$	$x_{4,10,18,19}$	$x_{2,5,6,9,11,14,15,16,17,20}$
Dq-VC-DRSA	I	$x_{1,3,7,8,12}$	$x_{4,18}$	$x_{2,5,6,9,10,11,13,14,15,16,17,19,20}$
	II	$x_{1,3,7,8,12,13,16}$	$x_{4,10,18,19}$	$x_{2,5,6,9,11,14,15,17,20}$

Table 6.8
Decision regions of upward union $Cl_3^{\succcurlyeq}$ in different models.

Model		Positive region	Negative region	Boundary region
DRSA		$\emptyset$	$x_{4,5,14,18}$	$x_{1,2,3,6,7,8,9,10,11,12,13,15,16,17,19,20}$
Rq-VC-DRSA		$\emptyset$	$x_{2,4,5,6,14,17,18,19}$	$x_{1,3,7,8,9,10,11,12,13,15,16,20}$
Aq-VC-DRSA		x_8	$x_{1,2,3,4,5,6,7,10,12,14,15,17,18,19}$	$x_{9,11,13,16,20}$
Dq-VC-DRSA	I	$\emptyset$	$x_{2,4,5,6,14,17,18,19}$	$x_{1,3,7,8,9,10,11,12,13,15,16,20}$
	II	x_8	$x_{1,2,3,4,5,6,7,10,12,14,15,17,18,19}$	$x_{9,11,13,16,20}$

The lower and upper approximations of downward union $Cl_2^{\preccurlyeq}$ in DqII-VC-DRSA are

$$\begin{aligned}\underline{R_A}^{II}_{(\alpha,k)}(Cl_2^{\preccurlyeq}) =&\{x_1, x_2, x_3, x_4, x_5, x_6, x_7,\\ &x_{10}, x_{12}, x_{14}, x_{15}, x_{17}, x_{18}, x_{19}\};\\ \overline{R_A}^{II}_{(\alpha,k)}(Cl_2^{\preccurlyeq}) =&\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_9, x_{10}, x_{11},\\ &x_{12}, x_{13}, x_{14}, x_{15}, x_{16}, x_{17}, x_{18}, x_{19}, x_{20}\}.\end{aligned}$$

The lower and upper approximations of upward union $Cl_2^{\succcurlyeq}$ in DqII-VC-DRSA are

$$\begin{aligned}\underline{R_A}^{II}_{(\alpha,k)}(Cl_2^{\succcurlyeq}) =&\{x_1, x_3, x_7, x_8, x_{12}, x_{13}, x_{16}\};\\ \overline{R_A}^{II}_{(\alpha,k)}(Cl_2^{\succcurlyeq}) =&\{x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_9,\\ &x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}, x_{17}, x_{20}\}.\end{aligned}$$

The lower and upper approximations of upward union $Cl_3^{\succcurlyeq}$ in DqII-VC-DRSA are

$$\begin{aligned}\underline{R_A}^{II}_{(\alpha,k)}(Cl_3^{\succcurlyeq}) &= \{x_8\};\\ \overline{R_A}^{II}_{(\alpha,k)}(Cl_3^{\succcurlyeq}) &= \{x_8, x_9, x_{11}, x_{13}, x_{16}, x_{20}\}.\end{aligned}$$

The related three-way decision regions of different upward and downward unions in different models are concluded in the Tables 6.5-6.8. We can make the comparisons between the proposed single-quantitative, double-quantitative variable consistency dominance-based rough set models and DRSA based on three disjoint decision regions.

Here, as $x_{12} \in Cl_2$, we also take $x_{12} \in Cl_2^{\succcurlyeq}$ for example. All of the customers x_1, x_3, x_5, x_7, x_8, x_9, x_{12}, x_{13}, x_{14}, x_{15}, x_{16} and x_{20} in $Cl_2^{\succcurlyeq}$ made an overall evaluation of *at least medium*. Let us analyze the limitation of DRSA and the advantage of the presented Sq-VC-DRSA and Dq-VC-DRSA models from this case study.

Limitation of DRSA:

It is easy to see that for the object x_{12} (see Table 6.2), $D^+_{R_A}(x_{12}) = \{x_7, x_8, x_{12}, x_{19}\}$. $x_{12} \in Cl_2$ is dominated by 4 objects x_7, x_8, x_{12}, x_{19}, however, one of these objects x_{19} belongs to the decision class Cl_1. It means that the evaluations for all the three considered condition attribute values from the customer x_{19} are not worse than the evaluations from the customer x_{12}, while the overall evaluation for the decision attribute value from x_{19} is worse than x_{12}, so x_{12} does not belong to the lower approximation of upward union $Cl_2^{\succcurlyeq}$. It is just for this reason x_{19} that object x_{12} is assigned into the boundary region from the definition of DRSA without any fault tolerance mechanism (see Table 6.7).

In the case study, we only considered 20 objects, and this inconsistency happens to the customer x_{12}. For the upward union $Cl_2^{\succcurlyeq}$, if the DRSA is used to analyze the consistency of elements in $Cl_2^{\succcurlyeq}$, a few elements (x_{19}) in the dominating set of x_{12} are worse than those in the decision class of x_{12}. Obviously, the reason for this is that the conditions for obtaining the upper and lower approximations of DRSA are too strict. When the number of objects in an ordered information becomes much larger, such inconsistency will often arise, resulting in more inconsistencies. After adding the fault-tolerant mechanism (adding the relative and absolute quantitative consistency levels), x_{12} belongs to the lower approximation, so that x_{12} is re-divided into positive region, which can overcome the limitation of DRSA to a certain extent and make the more reasonable decisions. Therefore, it is necessary to add the quantitative consistency levels to the DRSA.

Advantage of Sq-VC-DRSA and Dq-VC-DRSA:

For the relative quantitative consistency level $\alpha = 0.7$, it is easy to see that $|D^+_{R_A}(x_{12}) \cap Cl_2^{\succcurlyeq}|/|D^+_{R_A}(x_{12})| = 3/4 \geq \alpha$. x_{12} belongs to lower approximation of upward union $Cl_2^{\succcurlyeq}$ with relative quantitative consistency level $\alpha = 0.7$, then x_{12} is assigned into the positive region in Rq-VC-DRSA, DqI-VC-DRSA and DqII-VC-DRSA models. For the absolute quantitative consistency level $k = 2$, it is easy to see that $|D^+_{R_A}(x_{12})| - |D^+_{R_A}(x_{12}) \cap Cl_2^{\succcurlyeq}| = 1 \leq k$. x_{12} belongs to lower approximation of upward union $Cl_2^{\succcurlyeq}$ with absolute quantitative consistency level $k = 2$, then x_{12} is assigned into the positive region in Aq-VC-DRDA, DqI-VC-DRSA and DqII-VC-DRSA models.

Compared with DRSA, when we consider a certain level of quantitative information in the upper and lower approximations, we can avoid the inconsistency situation such as the object x_{12}. Thus, appropriately relaxing the restrictions between the dominating set (dominated set) and the upward union set or downward union set in DRSA provides us a more intuitive and acceptable semantic interpretation and enlarges the scope of application of DRSA.

7. Conclusions

The DRSA proposed by Greco et al. [7,8] has some effects in dealing with an ordered information system, however, if there is a hesitation about the value of the decision attributes in a given decision ordered information system, the disadvantage of DRSA will appear. The reason for the disadvantage of DRSA is that there is no fault-tolerance mechanism for the conditions given to the upper and lower approximations of DRSA, and the restrictions between the dominating set (the dominated set) and the upward union or downward union are too strict. Hence, Greco et al. [13] presented a VC-DRSA to improve the discussed disadvantage of DRSA. In order to further overcome the disadvantage of DRSA, we develop the relative and absolute quantitative consistency levels in an ordered information system when the upper and lower approximations of DRSA are required to contain relative and absolute quantitative information, and then construct the Sq-VC-DRSA models (including Rq-VC-DRSA and Aq-VC-DRSA) and Dq-VC-DRSA models. In particular, it could be seen from the definition of lower and upper approximations in Rq-VC-DRSA that the VC-DRSA presented by Greco et al. [12,13] is actually the Rq-VC-DRSA. In this paper, we explain the new concepts and models by using real-life case study, and discuss the decision regions of upward union and downward union of decision classes in different models. We also interpret these two kinds of quantitative consistency levels by analyzing the decision rules about the objects divided into different decision regions at different kind of quantitative consistency levels and by comparing them with DRSA. In the future work, some aspects of the Dq-VC-DRSA models deserve to be studied, including the uncertainty measure of the models and the attribute reduction method based on the two quantitative consistency levels.

Declaration of competing interest

There is no conflict of interest.

Acknowledgements

This work is supported by Natural Science Foundation of China (No. 11671109, No. 61772002, No. 61976245), and the Fundamental Research Funds for the Central Universities (No. XDJK2019B029, No. SWU119045, No. SWU119063).

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